Processus multistables : Propriétés locales et estimation

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"Le monde est la preuve que Dieu est un comité."
Bob Stokes

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Chapitre I

Introduction, notions générales

Les chroniques des phénomènes qui évoluent dans le temps sont la plupart du temps trop complexes pour être fidèlement représentées par des modèles mathématiques dont l'étude est abordable. On cherche alors en général à se concentrer sur un aspect du phénomène étudié, et à faire des simplifications qui permettent l'analyse. Il est souvent fructueux de se placer dans un cadre stochastique, c'est-à-dire de considérer le phénomène comme un processus aléatoire. Une hypothèse fréquente est alors de supposer que les accroissements de ce processus sont stationnaires et indépendants. Si on suppose de plus que les données présentent une forme d'invariance d'échelle, c'est-à-dire que les observations par exemple journalières ont la même loi de probabilité que les observations horaires ou mensuelles, modulo une renormalisation, alors on se place dans le cadre des *mouvements de Lévy stables*, sur lesquels il existe une littérature très riche. Ceux-ci sont principalement caractérisés par un exposant $\alpha \in (0, 2]$, l'exposant de stabilité, qui contrôle l'intensité des sauts dans les trajectoires. Un petit α signifie des sauts de tailles très dispersées, tandis que α proche de 2 se traduit par une trajectoire plus homogène. En ajustant α, on peut ainsi rendre compte de phénomènes ayant des allures différentes (voir Figure I.1). Les graphes de la figure I.1 pourraient ainsi par exemple représenter des approximations convenables de l'évolution de certains actifs financiers, comme ceux présentés sur la figure I.2.

Cependant, dans de nombreux cas, l'hypothèse de stationnarité des accroissements est trop loin de la réalité pour pouvoir être conservée. Pour rester dans le domaine financier, on s'attend intuitivement à ce que, suivant l'heure de la journée ou l'environnement économique, les fluctuations soudaines (les sauts) soient plus ou moins prononcées (voir par exemple Figure I.4). Il est donc important de pouvoir étendre les modèles de manière à accomoder cette variabilité supplémentaire. Malheureusement, la non-stationnarité n'est pas une propriété, et à ce titre, ne fournit pas de procédure opératoire pour définir des modèles plus généraux. Une façon de pallier ce problème, tout en conservant en partie certaines propriétés des processus stables, est de considérer des modèles qui, à chaque instant, "ressemblent" à un processus stable, mais dont l'exposant de stabilité varie au cours du temps : de même que la courbe représentative d'une fonction suffisamment régulière peut être approchée en chaque point par un segment de droite dont la pente est la dérivée de la fonction, on peut considérer des processus, dits *multistables*, qui sont en chaque point tangents, dans un sens qui sera précisé

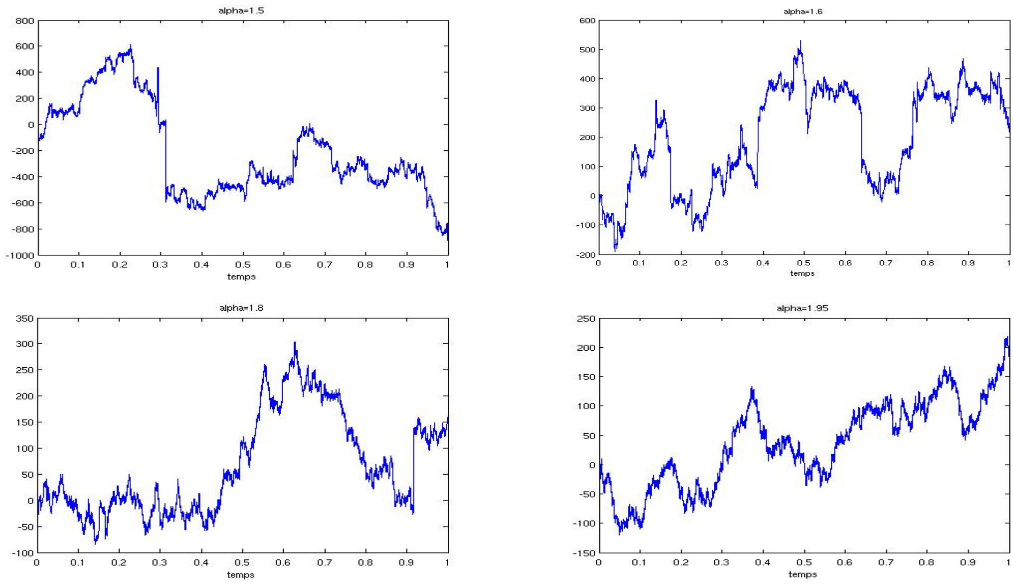


Figure I.1 – Réalisations de processus stables pour $\alpha = 1.5$, $\alpha = 1.6$, $\alpha = 1.8$ et $\alpha = 1.95$.

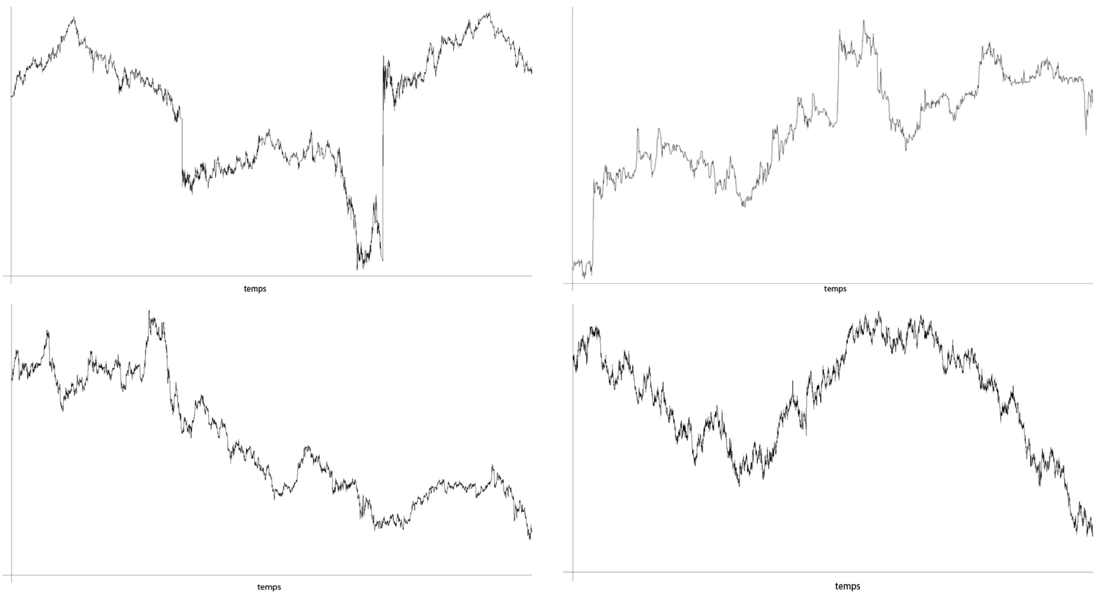


Figure I.2 – Cours financiers des entreprises CapGemini, Bonduelle, Dexia et France Télécom, dont les trajectoires présentent des sauts.

plus tard, à un processus stable. On saura alors modéliser des phénomènes qui, par moment, présentent une faible intensité de sauts, mais qui sont au contraire très erratiques à d'autres périodes. La figure I.3 montre des exemples de réalisations de tels processus. En comparant à la figure I.4, on voit que de tels modèles sont plus convaincants que ceux de la figure I.1.

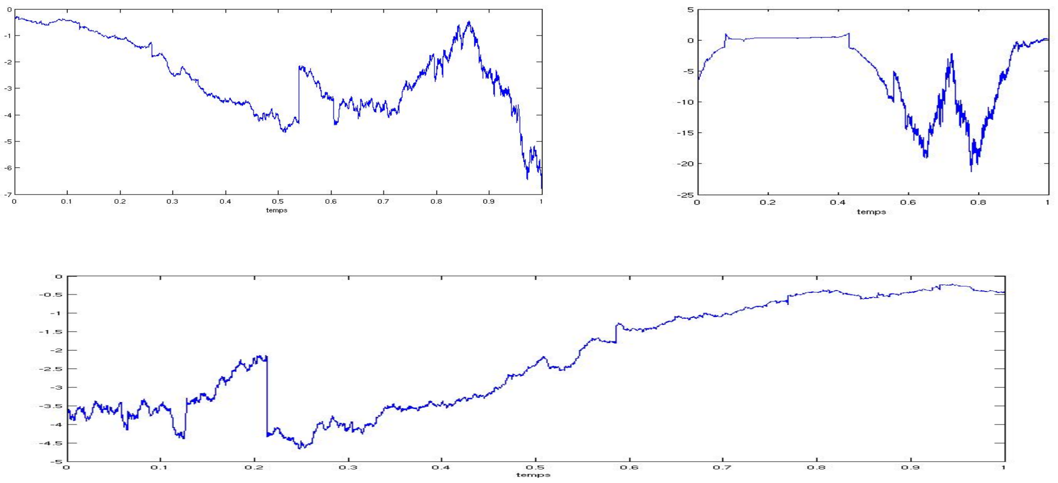


FIGURE I.3 – Réalisations de processus multistables présentant diverses évolutions de α.

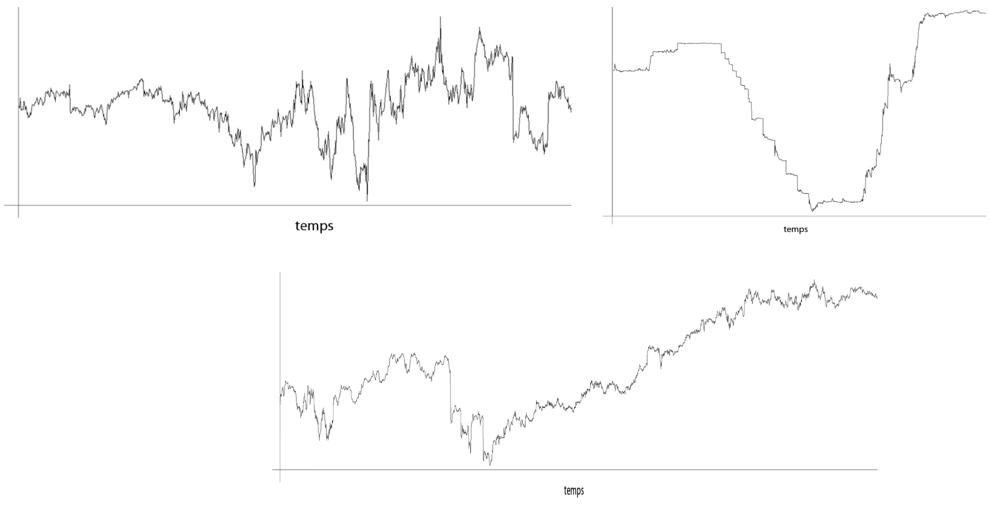


FIGURE I.4 – Cours financiers dont les accroissements sont clairement non stationnaires : l'intensité des sauts varie au cours du temps.

Outre l'intensité des sauts, on peut noter sur certaines chroniques de cours financiers que la "rugosité" locale des trajectoires évolue au cours du temps (voir Figure I.5). Mathématiquement, celle-ci peut-être mesurée par un exposant appelé exposant de Hölder. En s'éloignant un peu plus du cadre simple des mouvements de Lévy stables et en renonçant à l'hypothèse d'indépendance des accroissements, on peut construire des modèles où la rugosité varie au cours du temps, mais qui restent, à chaque instant, tangents à un processus à accroissements stationnaires. Un exemple de tel processus, dit *multistable multifractionnaire*, est présenté sur la figure I.6. On remarque que son aspect est similaire à celui du graphe de la figure I.5.

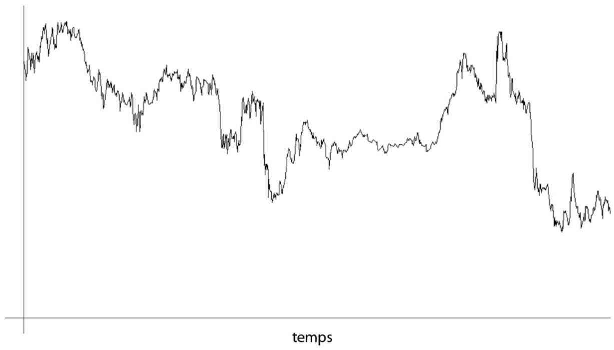


Figure I.5 – Cours financier dont la rugosité varie au cours du temps.

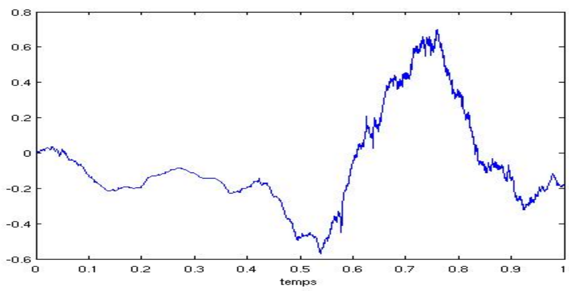


Figure I.6 – Réalisation d'un processus multistable multifractionnaire présentant des variations de la rugosité.

Le but de cette thèse est de définir et d'étudier de tels processus, en mettant l'accent sur certaines de leurs propriétés probabilistes et trajectorielles. Nous abordons aussi l'aspect statistique, dans l'optique de permettre l'application de nos modèles à des phénomènes réels.

I.1 Processus stables

On rappelle brièvement pour commencer une définition d'une variable aléatoire stable.

Définition I.1. Une variable aléatoire X est distribuée selon une loi stable s'il existe des paramètres $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\sigma \geq 0$ et $\mu \in \mathbb{R}$ tels que sa fonction caractéristique soit de la forme :

$$\mathbb{E}\left[e^{i\theta X}\right] = \begin{cases} \exp\left\{-\sigma^\alpha|\theta|^\alpha(1 - i\beta(\text{sign }\theta)\tan\frac{\pi\alpha}{2}) + i\mu\theta\right\} & \text{si } \alpha \neq 1 \\ \\ \exp\left\{-\sigma|\theta|(1 + i\beta\frac{2}{\pi}(\text{sign }\theta)\ln|\theta|) + i\mu\theta\right\} & \text{si } \alpha = 1 \end{cases}$$

où

$$\text{sign }\theta = \begin{cases} 1 & \text{si } \theta > 0, \\ 0 & \text{si } \theta = 0, \\ -1 & \text{si } \theta < 0. \end{cases}$$

On note alors, comme dans [49], $X \sim S_\alpha(\sigma, \beta, \mu)$.

Le paramètre α est l'indice de stabilité. Il caractérise la loi, avec les paramètres d'échelle σ, de biais β et de décalage μ.

On rappelle ensuite deux propriétés de moments pour ces variables stables (voir [49, Property 1.2.16 et Property 1.2.17]) :

Proposition I.1. Soit $X \sim S_\alpha(\sigma, \beta, \mu)$ avec $0 < \alpha < 2$. Alors

$$\begin{array}{lll} \mathsf{E}|X|^p < \infty & \text{pour tout} & 0 < p < \alpha, \\ \mathsf{E}|X|^p = \infty & \text{pour tout} & p \geq \alpha. \end{array}$$

Proposition I.2. Soit $X \sim S_\alpha(\sigma, \beta, 0)$ avec $0 < \alpha < 2$ et $\beta = 0$ dans le cas $\alpha = 1$. Alors, pour tout $p \in (0, \alpha)$, il existe une constante $c_{\alpha,\beta}(p)$ telle que

$$(\mathsf{E}\,|X|^p)^{1/p} = c_{\alpha,\beta}(p)\sigma. \tag{I.1}$$

L'expression de la constante est donnée dans [21] :

$$(c_{\alpha,\beta}(p))^p = \frac{2^{p-1}\Gamma(1-\frac{p}{\alpha})}{p\int_0^\infty \frac{\sin^2 u}{u^{p+1}}du}\left(1 + \beta^2\tan^2\frac{\alpha\pi}{2}\right)^{p/2\alpha}\cos\left(\frac{p}{\alpha}\arctan\left(\beta\tan\frac{\alpha\pi}{2}\right)\right). \tag{I.2}$$

Rappelons à présent qu'un processus $\{X(t) : t \in T\}$, où T est un sous-intervalle de $\mathbb{R}$, est appelé *α-stable* $(0 < \alpha \leq 2)$ si toutes ses lois finie-dimensionnelles sont α-stables (voir le travail encyclopédique sur les processus stables [49]). Les processus 2-stables sont les processus Gaussiens.

Nous allons donner maintenant trois représentations des processus α-stables. Dans toute la suite, nous supposerons $\beta = \mu = 0$. La plupart des résultats obtenus sont a priori généralisables au cas non symétrique, impliquant seulement des preuves plus complexes.

I.1.1 Représentation intégrale des processus stables

Beaucoup de processus stables admettent une représentation sous forme d'intégrale stochastique.

Soit $(E, \mathcal{E}, m)$ un espace de mesure σ-finie (m sera dans nos exemples la mesure de Lebesgue). En prenant m comme mesure de contrôle, on définit une mesure aléatoire α-stable M sur E telle que pour $A \in \mathcal{E}$, on ait $M(A) \sim S_\alpha(m(A)^{1/\alpha}, 0, 0)$ (comme $\beta = 0$, la variable est symétrique). Soit

$$\mathcal{F}_\alpha \equiv \mathcal{F}_\alpha(E, \mathcal{E}, m) = \{f : f \text{ est mesurable et } \|f\|_\alpha < \infty\},$$

où $\| \, \|_\alpha$ est la quasi-norme (ou norme pour $1 \leq \alpha \leq 2$) donnée par

$$\|f\|_\alpha = \left(\int_E |f(x)|^\alpha m(dx) \right)^{1/\alpha}. \tag{I.3}$$

L'intégrale stochastique de $f \in \mathcal{F}_\alpha(E, \mathcal{E}, m)$ par rapport à la mesure M existe alors (voir [49, Chapter 3]) avec

$$\int_E f(x) M(dx) \sim S_\alpha(\sigma_f, 0, 0), \tag{I.4}$$

où $\sigma_f = \|f\|_\alpha$. Pour l'étude des propriétés des variables et des processus stables, on peut se référer aux premiers chapitres de [49]. En particulier,

$$\mathsf{E}|I(f)|^p = \begin{cases} (c_{\alpha,\beta}(p))^p \|f\|_\alpha^p & (0 < p < \alpha) \\ \infty & (p \geq \alpha) \end{cases} \tag{I.5}$$

où $(c_{\alpha,\beta}(p))^p$ est défini en (I.2). On peut utiliser la norme $\| \, \|_\alpha$, entre autres pour démontrer la convergence en probabilité d'une suite d'intégrales stables en vertu de la proposition suivante :

Proposition I.3. Soit $X_j = \int_E f_j(x) M(dx)$, $j \in \mathbb{N}$ et soit l'intégrale $X = \int_E f(x) M(dx)$ où M est une mesure aléatoire α-stable de mesure de contrôle m. Alors, si $\xrightarrow{p}$ désigne la convergence en probabilité,

$$X_j \xrightarrow[j \to \infty]{p} X \Leftrightarrow \lim_{j \to \infty} \|f_j - f\|_\alpha = 0.$$

Pour terminer, on indique une formule de changement de variable, ou plus précisément de changement de mesure de contrôle. Elle permet notamment de passer du cas d'une mesure de contrôle finie au cas d'une mesure de contrôle σ-finie.

Proposition I.4 (Changement de variable)**.** Soit M_m et M_ν deux mesures α-stables avec $0<\alpha<2$, de mesures de contrôle respectives m et ν, vérifiant

$$\frac{m(dx)}{\nu(dx)} = (r(x))^\alpha, \quad x \in E, r(x) \geq 0.$$

Alors $\forall f \in \mathcal{F}^\alpha(E,m)$,

$$\int_E f(x) M_m(dx) \stackrel{d}{=} \int_E f(x) r(x) M_\nu(dx).$$

I.1.2 Représentation de Poisson d'une intégrale stable

On donne maintenant une autre représentation de l'intégrale stochastique stable à l'aide d'une mesure aléatoire de Poisson. Soit $(S,\mathcal{S},n)$ un espace mesuré et soit $\mathcal{S}_0 = \{A \in \mathcal{S} : n(A) < \infty\}$. En prenant n comme mesure de contrôle, on définit une mesure aléatoire de Poisson N sur $\mathcal{S}_0$ telle que pour $A \in \mathcal{S}_0$, on ait $N(A) \sim \mathcal{P}(n(A))$, i.e.

$$P(N(A)=k) = e^{-n(A)} \frac{(n(A))^k}{k!}, \quad k \geq 0.$$

Soit $S = E \times \mathbb{R}^\star$. Pour un ensemble mesurable $A \subset S$ et $g : A \to \mathbb{R}$ mesurable, on pose

$$\int_A g(x,u) N(dx,du) = \sum_{i=1}^\infty \mathbf{1}_A(X_i,U_i) g(X_i,U_i)$$

où $\{X_i,U_i\}$ est une réalisation des points de N.

On pose

$$n(dx,du) = \mathbb{E}N(dx,du) = m(dx) \frac{du}{|u|^{\alpha+1}},$$

où m est la mesure de contrôle d'une mesure aléatoire α-stable M.

On définit à présent comme dans [49] les intégrales stables à l'aide des intégrales de Poisson $\int_A g(x,u)N(dx,du)$. Soit $(E_i)_{i\geq 1}$ une partition de E en ensembles de mesures $m(E_i)$ finies (on suppose m σ-finie).

Théorème I.5 (Représentation de Poisson, cas symétrique)**.** *L'intégrale stochastique α-stable $\mathcal{I}(f) = \int_E f(x)M(dx)$ admet comme représentation (les limites sont au sens de la convergence presque sûre) :*

- $0<\alpha<1$ *:*

$$\mathcal{I}(f) \stackrel{d}{=} (2\alpha^{-1}\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2})^{-1/\alpha} \int_E \int_{\mathbb{R}^\star} f(x) u N(dx,du).$$

- $\alpha = 1$ *:*

$$\mathcal{I}(f) \stackrel{d}{=} \frac{1}{\pi} \lim_{\delta\to 0} \sum_{i=1}^\infty \int_{E_i} \int_{]-\delta,\delta[^c} f(x) u N(dx,du).$$

- $1 < \alpha < 2$:

$$\mathcal{I}(f) \stackrel{d}{=} \left(2\frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}(-\cos\frac{\pi\alpha}{2})\right)^{-1/\alpha} \times$$
$$\lim_{\delta\to 0}\sum_{i=1}^{\infty}\int_{E_i}\int_{]-\delta,\delta[^c} f(x)uN(dx,du).$$

I.1.3 Représentation de Ferguson - Klass - LePage

Nous présentons maintenant une représentation alternative d'une intégrale stochastique α-stable, à l'aide d'une somme convergente de temps d'arrivée d'un processus de Poisson. Ceci constitue un résultat très important pour la suite. Cette représentation est à la base de l'étude des processus multistables faite ici, aussi bien pour la définition des processus que pour l'établissement de leurs propriétés. Pour plus de détails sur cette représentation, on peut se référer par exemple à [4, 18, 32, 33, 48] et à [49, Theorem 3.10.1].

Soit $(E, \mathcal{E}, m)$ un espace de mesure finie, et U un intervalle ouvert de $\mathbb{R}$. On se donne $0 < \alpha < 2$ et une fonction f de l'espace $\mathcal{F}_\alpha(E, \mathcal{E}, m)$. On considère $(\Gamma_i)_{i\geq 1}$ une suite de temps d'arrivée d'un processus de Poisson d'intensité 1, i.e. $\Gamma_i = \sum\limits_{j=1}^{i} e_j$ où les e_j sont des variables aléatoires indépendantes identiquement distribuées de loi exponentielle de paramètre 1. On se donne également $(V_i)_{i\geq 1}$ une suite de variables aléatoires indépendantes identiquement distribuées de loi $\hat{m} = m/m(E)$ sur E, et $(\gamma_i)_{i\geq 1}$ une suite de variables aléatoires indépendantes identiquement distribuées de loi $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. On suppose finalement que les trois suites $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, et $(\gamma_i)_{i\geq 1}$ sont indépendantes.

On a alors le théorème de représentation en série de Ferguson-Klass-LePage de l'intégrale stochastique α-stable, énoncé ici comme dans [49, Theorem 3.10.1].

Théorème I.6 (Représentation de Ferguson-Klass-LePage, cas symétrique)**.** *On pose* $C_\eta = \left(\int\limits_0^\infty x^{-\eta}\sin(x)dx\right)^{-1}$ *et* $\mathcal{I}(f) = \int_E f(x)M(dx)$*, où* M *est une mesure aléatoire* α*-stable de mesure de contrôle* m*. La série* $(C_\alpha m(E))^{1/\alpha}\sum\limits_{i=1}^{\infty}\gamma_i\Gamma_i^{-1/\alpha}f(V_i)$ *est presque sûrement convergente et on a*

$$\mathcal{I}(f) \stackrel{d}{=} (C_\alpha m(E))^{1/\alpha}\sum_{i=1}^{\infty}\gamma_i\Gamma_i^{-1/\alpha}f(V_i).$$

Une preuve de ce théorème repose en partie sur le théorème des trois séries, que l'on peut trouver par exemple dans [42] :

Théorème I.7 (des trois séries)**.** *Soit* $(X_n)_{n\in\mathbb{N}}$ *une suite de variables aléatoires indépendantes. Pour que la série* $\sum\limits_{i=1}^{+\infty} X_i$ *converge presque sûrement, il est nécessaire que pour tout* $c > 0$*, les trois séries* $\sum_k P(|X_k| > c)$*,* $\sum_k Var(X_k\mathbf{1}_{|X_k|<c})$ *et* $\sum_k \mathsf{E}(X_k\mathbf{1}_{|X_k|<c})$ *convergent presque sûrement, et il est suffisant qu'elles convergent pour un* $c > 0$*.*

I.1.4 Exemples de processus stables

Dans la suite, M désignera une mesure aléatoire symétrique α-stable ($0 < \alpha < 2$) sur $\mathbb{R}$, de mesure de contrôle la mesure de Lebesgue $\mathcal{L}$.

Mouvement stable de Lévy

Soit

$$L_\alpha(t) := \int_0^t M(dz).$$

Le processus L_α vérifie les propriétés :

1. $L_\alpha(0) = 0$ presque sûrement.
2. L_α est à accroissements indépendants.
3. $L_\alpha(t) - L_\alpha(s) \sim S_\alpha((t-s)^{1/\alpha}, 0, 0)$.

Le processus L_α est un processus de Lévy. C'est le mouvement Brownien pour $\alpha = 2$. Il est $1/\alpha$-auto-similaire et à accroissements stationnaires et indépendants. C'est le seul processus α-stable vérifiant ces propriétés si $\alpha \in (0,1)$, ce qui n'est plus vrai pour $\alpha \geq 1$.

Mouvement Log-Fractionnaire Stable

Le *Mouvement Log-Fractionnaire Stable* est défini par

$$\Lambda_\alpha(t) = \int_{-\infty}^{\infty} (\log(|t-x|) - \log(|x|))\, M(dx) \quad (t \in \mathbb{R}).$$

Ce processus n'est bien défini que pour $\alpha \in (1,2]$ (le noyau n'appartient pas à $\mathcal{F}_\alpha$ pour $\alpha \leq 1$). Il est $1/\alpha$-auto-similaire à accroissements stationnaires.

Mouvement Linéaire Fractionnaire Stable

Ce processus est défini pour tout $t \in \mathbb{R}$ par :

$$L_{\alpha,H,b^+,b^-}(t) = \int_{-\infty}^{\infty} f_{\alpha,H}(b^+, b^-, t, x) M(dx)$$

où $H \in (0,1)$, $b^+, b^- \in \mathbb{R}$, et

$$\begin{aligned} f_{\alpha,H}(b^+, b^-, t, x) = b^+ &\left((t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha} \right) \\ &+ b^- \left((t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha} \right). \end{aligned}$$

L_{α,H,b^+,b^-} est un processus H-auto-similaire à accroissements stationnaires. Quand $b^+ = b^- = 1$, il est appelé Mouvement Linéaire Fractionnaire α-Stable Bien Equilibré, et est noté $L_{\alpha,H}$. On a

$$L_{\alpha,H}(t) = \int_{-\infty}^{\infty} (|t-x|^{H-1/\alpha} - |x|^{H-1/\alpha}) M(dx). \tag{I.6}$$

Dans le cas $\alpha = 2$, on retrouve le mouvement Brownien fractionnaire.

S. Stoev et M.S. Taqqu ont introduit en 2004 une version multifractionnaire du processus $L_{\alpha,H}$ (voir [54]), reprenant ainsi une construction analogue à celle de R.F. Peltier et J. Lévy-Véhel dans [40] pour définir une version multifractionnaire du mouvement Brownien fractionnaire. Les propriétés trajectorielles du processus stable $L_{\alpha,H(t)}(t)$ où H est une fonction et

$$L_{\alpha,H(t)}(t) = \int_{-\infty}^{\infty} (|t-x|^{H(t)-1/\alpha} - |x|^{H(t)-1/\alpha}) M(dx), \tag{I.7}$$

sont étudiées dans [55].

Processus d'Ornstein-Uhlenbeck Rétrograde Stable

Soit $\lambda > 0$. Le processus stationnaire défini par

$$Y(t) = \int_{t}^{\infty} \exp(-\lambda(x-t)) M(dx), \quad (t \in \mathbb{R}),$$

est appelé *Processus d'Ornstein-Uhlenbeck Rétrograde Stable.*

Dans la suite, nous étudierons des "versions multistables" des processus introduits dans cette partie.

I.2 Régularité locale

L'étude de la régularité locale est utile dans beaucoup de domaines, comme les équations aux dérivées partielles, la mécanique des fluides, l'analyse financière, l'analyse des trafics, ou encore le traitement du signal. En turbulence, l'énergie dissipée est directement liée à la structure des singularités du flux [19]. En analyse de données de trafics ou financières, la régularité locale donne une mesure de l'erraticité des enregistrements autour d'un point donné, permettant ainsi un meilleur contrôle de ces données [2, 46]. Enfin, dans le domaine du traitement du signal, la régularité locale est parfois plus pertinente que par exemple l'amplitude, pour la segmentation ou la détection [9, 35, 41].

Il est donc important de définir de manière pertinente une mesure de la régularité locale. Beaucoup d'exposants de régularité ont été proposés et étudiés ces dernières années. On peut se référer par exemple à [10, 24, 25, 28, 38, 51, 52]. Dans cette thèse, nous nous intéresserons uniquement à l'exposant de Hölder ponctuel $\mathcal{H}$ et à l'exposant de Hölder local h_l. La question qui se pose alors est de savoir quelles sont les relations entre ces différents exposants. Par exemple, on montre dans [10] et [24] que la fonction de Hölder ponctuelle, c'est-à-dire la fonction qui à un point t associe son exposant de Hölder ponctuel $\mathcal{H}(t)$, est la limite inférieure d'une suite de fonctions continues. De même, la fonction de Hölder locale doit être une fonction semi-continue inférieurement ([20, 51]). De plus il y a quelques relations entre les exposants. Par exemple, les fonctions de Hölder ponctuelle et locale doivent coïncider sur un ensemble non dénombrable de points [51], et on a la relation $h_l \leq \mathcal{H}$.

Définissons à présent ces exposants de régularité classiques.

I.2.1 Définitions d'exposants de régularité

La plupart du temps, on mesure la régularité locale en utilisant l'exposant de Hölder ponctuel $\mathcal{H}$. On rappelle ici sa définition.

Définition I.2. Soit $x_0 \in \mathbb{R}$, et $s \in \mathbb{R}$ tel que $s > -1$. Une fonction $f : \mathbb{R} \to \mathbb{R}$ appartient à l'espace $C^s_{x_0}$ si et seulement s'il existe une constante C et un polynôme P de degré au plus $[s]$ [1] tels qu'au voisinage de x_0

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^s.$$

L'exposant de Hölder *ponctuel* de f au point x_0, noté $\mathcal{H}(x_0)$, est alors défini par

$$\mathcal{H}(x_0) = \sup\{s : f \in C^s_{x_0}\}. \tag{I.8}$$

Dans le cas d'une fonction non différentiable, la définition (I.8) devient

$$\mathcal{H}(t) = \sup\{\gamma : \lim_{r\to 0} \frac{|f(t+r) - f(t)|}{|r|^\gamma} = 0\}.$$

Dans certains cas, la seule connaissance de $\mathcal{H}$ ne donne pas assez d'informations, et se révèle même parfois hors de propos. En traitement du signal, par exemple, on doit souvent faire agir sur les signaux des opérateurs pseudo-différentiels, tels que la transformée de Hilbert. Le problème est alors que l'exposant $\mathcal{H}$ n'est pas stable sous l'action de ces opérateurs, donc on ne peut prédire quel sera l'exposant de Hölder ponctuel du signal transformé. On dispose de plusieurs manières pour apporter de l'information supplémentaire et ainsi obtenir une description plus riche.

Une possibilité consiste à utiliser l'exposant de Hölder local h_l. Pour définir h_l, on pose d'abord $h_l\left(f, x_0, \eta\right) = \sup\left\{\alpha : f \in C^\alpha\left(B\left(x_0, \eta\right)\right)\right\}$, où $C^\alpha(E)$ est l'espace de Hölder global usuel sur E et $B(x_0, \eta)$ est la boule ouverte de centre x_0 et de rayon η. $h_l\left(f, x_0, \eta\right)$ est alors une fonction décroissante de η, et on pose :

Définition I.3. (voir [20])
Soit $f : \mathbb{R} \to \mathbb{R}$ une fonction. L'exposant de Hölder *local* de f au point x_0 est le nombre réel :

$$h_l\left(x_0\right) = \lim_{\eta\to 0} h_l\left(f, x_0, \eta\right).$$

Contrairement à $\mathcal{H}$, h_l est stable sous l'action d'opérateurs pseudo-différentiels. Une étude complète de h_l et de ses liens avec $\mathcal{H}$ est faite dans [51].

I.2.2 Exemples

Dans le cas de fonctions singulières telles que $f(x) = |x|^\gamma$, $\mathcal{H}(0) = h_l(0) = \gamma$ et h_l n'apporte pas de nouvelle information par rapport à $\mathcal{H}$. C'est aussi le cas pour des fonctions partout irrégulières avec une fonction de Hölder ponctuelle régulière comme dans le cas de la fonction de Weierstrass.

1. $[s]$ désigne la partie entière de s.

La fonction la plus simple telle que l'exposant local apporte de l'information est la fonction $f(x) = |x|^\gamma \sin \frac{1}{|x|^\beta}$ où $\gamma > 0$, $\beta \geq 0$. Dans ce cas, on montre que $\mathcal{H}(0) = \gamma$ et $h_l(0) = \frac{\gamma}{1+\beta}$.

Concernant les processus stochastiques, on donne ici les exemples du Mouvement Brownien Fractionnaire et du Mouvement Brownien Multifractionnaire définis par les relations (I.6) et (I.7) pour $\alpha = 2$. On a tout d'abord pour le Mouvement Brownien Fractionnaire un résultat que l'on peut retrouver par exemple dans [12] :

Proposition I.8. Soit $L_{2,H}$ le processus défini par la relation (I.6). Les exposants de Hölder satisfont la relation, pour presque toute trajectoire,

$$\forall t \in \mathbb{R}^+, \quad \mathcal{H}(t) = h_l(t) = H.$$

Pour la version multifractionnaire du processus $L_{2,H}$, le Mouvement Brownien Multifractionnaire défini par la relation (I.7), les exposants sont étudiés dans [23]. On peut dans certains cas expliciter complètement les exposants.

Proposition I.9. Soit U un intervalle ouvert de $\mathbb{R}$. Soit H une fonction Höldérienne telle que son exposant de Hölder local $\tilde{h}$ vérifie la relation $\forall t \in U$, $H(t) < \tilde{h}(t)$. Soit Y le processus défini par (I.7).

Alors, presque sûrement, pour tout $t \in U$,

$$\mathcal{H}(t) = h_l(t) = H(t).$$

L'exposant local n'est cependant pas toujours égal à l'exposant ponctuel. Un exemple classique où les exposants sont différents provient de la classe des processus de Lévy. Pour un tel processus Z symétrique défini sur $\mathbb{R}$, d'exposant caractéristique

$$\Phi(u) = \int_{\mathbb{R}} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \nu(dx),$$

où ν est une mesure borélienne telle que $\int_{\mathbb{R}} \frac{x^2}{1+x^2} \nu(dx) < +\infty$, on introduit comme dans [6] l'indice β par la relation

$$\beta = \inf_{\alpha > 0} \left\{ \int_{|x| \leq 1} |x|^\alpha d\nu(x) < +\infty \right\}.$$

Un résultat de [45] montre alors que, pour un point t donné, l'exposant de Hölder ponctuel $\mathcal{H}(t)$ de Z vérifie la relation, presque sûrement, $\mathcal{H}(t) = \frac{1}{\beta}$. Pour le Mouvement de Lévy symétrique stable L_α, on a $\beta = \alpha$, d'où la relation, pour tout $t \in (0, 1)$, presque sûrement,

$$\mathcal{H}(t) = \frac{1}{\alpha}.$$

On sait de plus que l'exposant local est différent, puisque $h_l \equiv 0$.

Nous démontrerons un résultat analogue concernant le Mouvement de Lévy multistable dans le chapitre IV. Pour cela nous utiliserons à plusieurs reprises le lemme suivant, que l'on peut retrouver par exemple dans [34, Lemme 1.5] :

Lemme I.10. Soit $(X_i)_{i\leq N}$ des variables aléatoires indépendantes centrées. On suppose que $\|X_i\|_\infty < +\infty$ et on pose $a = \left(\sum_{i=1}^{N} \|X_i\|_\infty^2\right)^{1/2}$. Alors, pour tout $t > 0$,

$$\mathsf{P}\left(|\sum_{i=1}^{N} X_i| > t\right) \leq 2e^{-t^2/2a^2}.$$

I.3 Processus localisables

Les processus localisables sont utiles en théorie et en pratique. En effet, ils fournissent un moyen efficace de contrôler des propriétés locales variant au cours du temps telles que l'exposant de Hölder ponctuel ou encore l'intensité des sauts. Dans le premier cas, on parle de processus *multifractionnaires*, et dans le second cas de processus *multistables*.

I.3.1 Définition et propriétés

On donne à présent la définition d'un processus localisable [13, 14] :

Définition I.4. On dit que $Y = \{Y(t), t \in \mathbb{R}\}$ est $h(u)$-localisable au point $u \in \mathbb{R}$ s'il existe un processus non trivial $Y'_u = \{Y'_u(t), t \in \mathbb{R}\}$ tel que

$$\lim_{r\to 0} \frac{Y(u+rt) - Y(u)}{r^{h(u)}} = Y'_u(t), \tag{I.9}$$

où la convergence est au sens des lois finies-dimensionnelles.

Quand le processus Y'_u existe, il est appelé *forme locale*. En général, il varie avec le point u. Si Y et Y'_u ont des versions dans $C(\mathbb{R})$ (l'espace des fonctions continues sur $\mathbb{R}$) ou $D(\mathbb{R})$ (l'espace des fonctions càdlàg sur $\mathbb{R}$), et si la convergence est en loi pour la métrique appropriée, on dit que Y est *fortement localisable* au point u. Cette notion est proche de celle des processus localement asymptotiquememt auto-similaires, décrite par exemple dans [7].

Les processus localisables les plus simples sont les processus auto-similaires à accroissements stationnaires.

Proposition I.11. (voir [13]) Soit $\{Y(t), t \in \mathbb{R}\}$ un processus h-auto-similaire à accroissements stationnaires. Alors Y est h-localisable pour tout $u \in \mathbb{R}$, avec $Y'_u = Y$. Si de plus Y admet une version dans $C(\mathbb{R})$ ou $D(\mathbb{R})$, alors Y est fortement localisable pour tout $u \in \mathbb{R}$.

La proposition suivante montre que le produit par une fonction Höldérienne n'influe pas sur la localisabilité (voir [16]).

Proposition I.12. Soit U un intervalle et u un point intérieur. On suppose que $\{Y(t), t \in U\}$ est h-localisable en u. Soit $a : U \to \mathbb{R}$ une fonction η-Höldérienne :

$$|a(t) - a(t')| \leq c|t-t'|^\eta \quad (t, t' \in U), \eta > h.$$

Alors $aY = \{a(t)Y(t), t \in U\}$ est h-localisable avec $(aY)'_u = a(u)Y'_u$.

Ainsi la classe des processus localisables comprend par exemple le Mouvement stable de Lévy L_α, le Mouvement Linéaire Fractionnaire Stable Bien Equilibré $L_{\alpha,H}$, ou encore le Mouvement Log-Fractionnaire Stable et le Processus d'Ornstein-Uhlenbeck Rétrograde Stable (pour ce dernier, la localisabilité est prouvée dans [15]).

Un autre exemple classique est le mouvement Brownien multifractionnaire (mBm) Y, qui ressemble, au voisinage du point u, au mouvement Brownien fractionnaire de paramètre $h(u)$ (noté $B_{h(u)}$), c'est-à-dire

$$\lim_{r\to 0} \frac{Y(u+rt)-Y(u)}{r^{h(u)}} \stackrel{d}{=} B_{h(u)}(t). \tag{I.10}$$

Une généralisation du mBm, en remplaçant la mesure gaussienne par une mesure α-stable conduit au Mouvement Linéaire Stable Multifractionnaire défini par (I.7). Sous certaines conditions sur la fonction h, K. Falconer et J. Lévy-Véhel ont obtenu la localisabilité du processus, d'après le théorème suivant de [16] :

Théorème I.13 (Mouvement Linéaire Stable Multifractionnaire)**.** *Soit U un intervalle fermé avec u point intérieur. Soit $0<\alpha<2$ et $H : U \to (0,1)$. Soit $Y = \{L_{\alpha,H(t)}(t), t \in U\}$ défini par (I.7).*
a) On suppose que H vérifie une condition η-Höldérienne en u :

$$|H(v) - H(u)| \le k|v-u|^\eta \quad (v \in U)$$

où $H(u) < \eta \le 1$. Alors Y est $H(u)$-localisable en u avec pour forme locale $Y'_u = L_{\alpha,H(u)}$.
b) Si $1<\alpha<2$ et H est différentiable avec $1/\alpha < H(u) < 1$ et

$$|H'(v) - H'(v')| \le k|v-v'|^\eta \quad (v, v' \in U)$$

où $1/\alpha < \eta \le 1$, alors Y est fortement $H(u)$-localisable en u avec pour forme locale $Y'_u = L_{\alpha,H(u)}$.

I.3.2 Forme locale prescrite de processus localisables

On veut construire ici des processus localisables de forme locale donnée. La h-forme locale Y'_u au point u, si elle existe, doit être h-auto-similaire, c'est-à-dire $Y'_u(rt) \stackrel{d}{=} r^h Y'_u(t)$ pour $r > 0$. De plus, d'après [13, 14], sous certaines hypothèses générales, Y'_u doit également être à accroissements stationnaires pour presque tout u. Ainsi les formes locales typiques sont auto-similaires à accroissements stationnaires, c'est-à-dire $r^{-h}(Y'_u(u+rt) - Y'_u(u)) \stackrel{d}{=} Y'_u(t)$ pour tout u et $r > 0$.

L'ensemble des processus auto-similaires à accroissements stationnaires comprend, par exemple, le Mouvement Brownien fractionnaire, le Mouvement Linéaire Fractionnaire Stable et le Mouvement Stable de Lévy (voir [12, 49]).

Dans [16], on construit des processus de forme locale prescrite en juxtaposant des processus localisables connus : soit U un intervalle et u un point intérieur de U. Soit $\{X(t,v) : (t,v) \in U \times U\}$ un champ aléatoire et Y le processus diagonal $Y = \{X(t,t) : t \in U\}$. De manière à ce que Y et $X(\cdot, u)$ aient la même forme locale au point u, c'est-à-dire $Y'_u(\cdot) = X'_u(\cdot, u)$ où $X'_u(\cdot, u)$ est la forme locale $X(\cdot, u)$ au point u, une condition suffisante est donnée par

$$\lim_{r \to 0} \frac{X(u + rt, u + rt) - X(u, u)}{r^h} = X'_u(t, u) \tag{I.11}$$

où la convergence est en lois finies-dimensionnelles.

Cette approche permet de construire facilement des processus localisables à partir de processus que l'on sait localisables. On peut l'utiliser en particulier pour des champs $X(t, v)$ tels que pour chaque v, le processus $X(\cdot, v)$ est auto-similaire à accroissements stationnaires. Ainsi nous construisons au Chapitre III des processus localisables de manière analogue à celle de [16], en utilisant le champ aléatoire $\{X(t, v), (t, v) \in \mathbb{R}^2\}$, où t est le temps, et où le processus $t \mapsto X(t, v)$ est localisable pour tout v. Ce champ permet ainsi de contrôler la forme locale du processus diagonal $Y = \{X(t, t) : t \in \mathbb{R}\}$. Par exemple, dans le cas du mBm, X sera un champ de mouvements Browniens fractionnaires, *i.e.* $X(t, v) = B_{h(v)}(t)$, où h est une fonction lisse de v à valeurs dans $(0, 1)$. Cela correspond à l'approche originale utilisée pour étudier le mBm dans [1].

D'un point de vue heuristique, prendre la diagonale d'un tel champ stochastique permet de construire un nouveau processus dont la forme locale dépend du paramètre t. Nous utiliserons des champs aléatoires $\{X(t, v) : (t, v) \in \mathbb{R}^2\}$ tels que pour chaque v, la forme locale $X'_v(\cdot, v)$ de $X(\cdot, v)$ au point v soit la forme locale désirée, c'est-à-dire celle de Y au point v, Y_v.

Des critères généraux garantissant le transfert de localisabilité de $X(\cdot, v)$ vers $Y = \{X(t, t) : t \in \mathbb{R}\}$ sont obtenus dans [16]. Dans la suite, nous utiliserons le critère suivant :

Théorème I.14. *Soit U un intervalle et u un point intérieur. On suppose que pour un certain $0 < h < \eta$, le processus $\{X(t, u), t \in U\}$ est h-localisable au point $u \in U$ de forme locale $X'_u(\cdot, u)$ et*

$$\mathsf{P}(|X(v, v) - X(v, u)| \geq |v - u|^\eta) \to 0 \tag{I.12}$$

quand $v \to u$. Alors $Y = \{X(t, t) : t \in U\}$ est h-localisable au point u avec $Y'_u(\cdot) = X'_u(\cdot, u)$.

I.4 Processus multistables

Nous présentons ici brièvement deux approches existantes pour la construction de processus multistable.

La méthode de [16] consiste à utiliser la construction de processus localisables de forme locale prescrite de la section I.3.2, avec pour forme locale prescrite un processus stable. On note $\mathcal{L}$ la mesure de Lebesgue sur $\mathbb{R}$, et on se donne un processus de Poisson Π sur $\mathbb{R}^2$ d'intensité $\mathcal{L}^2$. Il s'agit donc d'un ensemble dénombrable de points aléatoires de $\mathbb{R}^2$, tel qu'en notant $N(A)$ le nombre de points de Π appartenant à A un ensemble mesurable de $\mathbb{R}^2$, $N(A)$ est une variable aléatoire de Poisson d'intensité $\mathcal{L}^2(A)$.

On définit ensuite l'espace $\mathcal{F}_{a,b}$ pour $0 < a < b < 2$ par

$$\mathcal{F}_{a,b} = \{f : f \text{ est mesurable et } \|f\|_{a,b} < +\infty\}$$

où

$$\|f\|_{a,b} = \left(\int_{\mathbb{R}} |f(x)|^a dx\right)^{1/a} + \left(\int_{\mathbb{R}} |f(x)|^b dx\right)^{1/b}. \tag{I.13}$$

On se donne α une fonction de classe C^1 définie sur U intervalle ouvert de $\mathbb{R}$, et $f(t,u,.)$ une famille de fonctions telles que, pour tout $(t,v) \in U^2$, $f(t,v,.) \in \mathcal{F}_{a,b}$. On définit le champ

$$X(t,v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t,v,\mathsf{X})\mathsf{Y}^{<-1/\alpha(v)>},$$

où $\mathsf{Y}^{<-1/\alpha(v)>} = \mathrm{sign}(\mathsf{Y})|\mathsf{Y}|^{-1/\alpha(v)}$. Un processus multistable est alors défini comme le processus diagonal

$$Y(t) = X(t,t). \tag{I.14}$$

On trouve dans [16] des critères sur les fonctions f et α pour obtenir des processus multistables Y localisables, ainsi que des critères de localisabilité forte. On rappelle ici ceux permettant d'obtenir la localisabilité simple.

Théorème I.15. *Soit U un intervalle fermé de $\mathbb{R}$, u un point intérieur de U et $0 < a < b < 2$. Soit X le champ aléatoire défini par*

$$X(t,v) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} f(t,v,\mathsf{X})\mathsf{Y}^{<-1/\alpha(v)>}, \quad (t, v \in U)$$

où $f(t,v,.) \in \mathcal{F}_{a,b}$ sont des fonctions conjointement mesurables et $\alpha : U \to (a,b)$.

On suppose que $X(.,u)$ est h-localisable au point u pour $h > 0$. On suppose que $\sup_{t\in U} \|f(t,v,.)\|_{a,b} < \infty$, et que pour un certain $\eta > h$,

$$|\alpha(v) - \alpha(u)| \leq k_1 |v-u|^\eta \quad (v \in U),$$

et

$$\|f(t,v,.) - f(t,u,.)\|_{a,b} \leq k_2 |v-u|^\eta \quad (t, v \in U).$$

Alors le processus $Y = \{X(t,t) : t \in U\}$ est h-localisable au point u de forme locale $Y'_u(.) = X'_u(.,u)$.

On dispose également d'une autre notion de processus multistable, introduite dans [17] à l'aide de fonctions caractéristiques.

On se donne une fonction $\alpha : \mathbb{R} \to [a,b]$ Lebesgue mesurable pour $0 < a \leq b \leq 2$. Pour $(f_1, f_2, ..., f_d) \in \mathcal{F}^d_{a,b}$ et $(\theta_1, \theta_2, ..., \theta_d) \in \mathbb{R}^d$, on pose

$$\Phi_{f_1,...,f_d}(\theta_1, ..., \theta_d) = \exp\left(-\int \left|\sum_{j=1}^d \theta_j f_j(x)\right|^{\alpha(x)} dx\right). \tag{I.15}$$

On montre alors que $\Phi_{f_1,...,f_d}$ définit bien une fonction caractéristique, et en appliquant le théorème d'extension de Kolmogorov, on définit un processus $\{I_m(f) : f \in \mathcal{F}_{a,b}\}$ appelé

Intégrale multistable de f. Ceci conduit à une autre définition des processus multistables, non équivalente à (I.14). En effet, pour un même noyau f, il est facile de voir que $I_m(f)$ et Y n'auront pas les mêmes lois. On donne également des critères de localisabilité de ces intégrales multistables dans [17].

Théorème I.16. *Soit $\alpha : \mathbb{R} \to [a, b] \subset (0, 2)$ une fonction continue et $Y(t) = I_m(f(t, .))$. On suppose que $f(t, .) \in \mathcal{F}_{a,b}$ pour tout t et qu'il existe une fonction h mesurable telle que pour tout t, $h(t, .) \in \mathcal{F}_{a,b}$ et*

$$\lim_{r \to 0} \| \frac{f(u + rt, u + r.) - f(u, u + r.)}{r^{H-1/\alpha(u+r.)}} - h(t, .)\|_{a,b} = 0.$$

Alors Y est H-localisable au point u de forme locale $Y'_u = \left\{\int h(t, z) M_{\alpha(u)}(dz)\right\}$ où $M_{\alpha(u)}$ est une mesure $\alpha(u)$-stable.

Nous travaillerons ici dans le même esprit que celui de [16]. Dans le chapitre II, nous utiliserons la représentation de Poisson des processus multistables, avant de proposer dans le chapitre III une définition alternative basée sur la représentation de Ferguson - Klass - LePage des processus stables.

I.5 Présentation des travaux de la thèse

Nous avons effectué jusqu'à présent quelques rappels concernant l'étude des processus stochastiques, ainsi que les définitions et les principaux résultats utilisés par la suite. L'objectif de cette thèse est d'étudier une nouvelle classe de processus, celle des processus multistables définis à partir d'une représentation de Ferguson - Klass - LePage.

Le chapitre II, extrait d'un article écrit en collaboration avec K. Falconer et J. Lévy-Véhel [15], présente des critères de localisabilité pour des moyennes mobiles stables et multistables définies par la représentation de Poisson. On donne ainsi des critères sur le noyau ou des critères sur sa transformée de Fourier, pour obtenir des moyennes mobiles localisables.

Proposition I.17. Soit $0 < \alpha \leq 2$, $g \in \mathcal{F}_\alpha$ et M une mesure aléatoire symétrique α-stable sur $\mathbb{R}$ de mesure de contrôle $\mathcal{L}$. Soit Y le processus à moyenne mobile

$$Y(t) = \int g(t - x) M(dx) \quad (t \in \mathbb{R}). \tag{I.16}$$

S'il existe $c_0^+, c_0^-, \gamma, a, c, \eta \in \mathbb{R}$ avec $c > 0$, $\eta > 0$ et $0 < \gamma + 1/\alpha < a \leq 1$ tels que

$$\frac{g(r)}{r^\gamma} \to c_0^+ \text{ et } \frac{g(-r)}{r^\gamma} \to c_0^-$$

quand $r \searrow 0$ et

$$|g(u + h) - g(u)| \leq c|h|^a |u|^{\gamma - a} \quad (u \in \mathbb{R}, |h| < \eta), \tag{I.17}$$

alors Y est $(\gamma + 1/\alpha)$-localisable au point $u \in \mathbb{R}$, avec pour forme locale

$$(a) \quad Y'_u = L_{\alpha, \gamma+1/\alpha, c_0^+, c_0^-} \quad \text{si } \gamma \neq 0,$$

$$(b) \quad Y'_u = (c_0^+ - c_0^-) L_\alpha \quad \text{si } \gamma = 0.$$

Si, de plus, $\gamma > 0$ et $0 < \alpha < 2$ alors Y admet une version dans $C(\mathbb{R})$ et est fortement localisable.

On peut aussi donner des conditions suffisantes de localisabilité portant sur la transformée de Fourier du noyau :

Proposition I.18. Soit $1 \leq \alpha \leq 2$, et Y défini par (I.16). S'il existe $l = l_1 + il_2 \in \mathbb{C}^*$, $\gamma \in (-\frac{1}{\alpha}, 1 - \frac{1}{\alpha})$, $a \in (0, 1 - (\gamma + \frac{1}{\alpha}))$ et $K \in L^p(\mathbb{R})$ avec $p \in [1, 1/(\gamma + \frac{1}{\alpha} + a))$, tels que pour presque tout $\xi > 0$,

$$\xi^{\gamma+1}\widehat{g}(\xi) = l + \frac{1}{\xi^a}\widehat{K}(\xi), \tag{I.18}$$

alors Y est $(\gamma + 1/\alpha)$-localisable pour tout point $u \in \mathbb{R}$, avec pour forme locale

$$(a) \quad Y'_u = L_{\alpha,\gamma+1/\alpha,b^+,b^-} \quad \text{si } \gamma \neq 0,$$

$$(b) \quad Y'_u = \tfrac{1}{\pi} l_1 Z_\alpha + l_2 L_\alpha \quad \text{si } \gamma = 0,$$

où

$$b^+ = \frac{1}{2\Gamma(\gamma+1)}\left(\frac{l_1}{\cos(\pi(\gamma+1)/2)} - \frac{l_2}{\sin(\pi(\gamma+1)/2)}\right),$$

$$b^- = \frac{1}{2\Gamma(\gamma+1)}\left(\frac{l_1}{\cos(\pi(\gamma+1)/2)} + \frac{l_2}{\sin(\pi(\gamma+1)/2)}\right).$$

Le cas de moyennes mobiles multistables $Y(t) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} g(\mathsf{X}-t)\mathsf{Y}^{<-1/\alpha(t)>}$, $(t \in \mathbb{R})$ est également traité dans ce chapitre avec le théorème II.9, que l'on peut appliquer, par exemple, au processus de Ornstein-Uhlenbeck rétrograde.

Proposition I.19 (Processus d'Ornstein-Uhlenbeck rétrograde multistable)**.** Soit $\lambda > 0$ et $\alpha : \mathbb{R} \to (1, 2)$ une fonction continument différentiable. Soit

$$Y(t) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi, \mathsf{X}\geq t} \exp(-\lambda(\mathsf{X}-t))\mathsf{Y}^{<-1/\alpha(t)>} \quad (t \in \mathbb{R}).$$

Alors Y est $1/\alpha(u)$-localisable en tout point $u \in \mathbb{R}$ avec $Y'_u = c(\alpha(u))^{-1} L_{\alpha(u)}$, où L_α est le Mouvement de Lévy α-stable.

La dernière partie du chapitre II aborde la question de la simulation de tels processus. On propose une méthode de simulation, ainsi qu'un contrôle de l'erreur d'approximation dans le cas stable.

Les chapitres III et IV sont des extraits d'articles écrits en collaboration avec J. Lévy-Véhel, respectivement [29] et [30].

Dans le chapitre III, nous considérerons une construction des processus multistables alternative à celle introduite dans [16], basée sur la représentation de Ferguson-Klass-LePage des processus stables.

On considère $(E, \mathcal{E}, m)$ un espace de mesure finie, et U un intervalle ouvert de $\mathbb{R}$. On se donne $(\Gamma_i)_{i\geq1}$ une suite de temps d'arrivée d'un processus de Poisson d'intensité 1, $(\gamma_i)_{i\geq1}$

de variables aléatoires indépendantes identiquement distribuées selon la loi $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$, et $(V_i)_{i\geq 1}$ un suite de variables aléatoires indépendantes et identiquement distribuées selon la loi $\hat{m} = m/m(E)$ sur E. On supposera que les trois suites $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, et $(\gamma_i)_{i\geq 1}$ sont indépendantes.

Lorsque la mesure n'est pas finie, on ne peut considérer $\hat{m}$. Cependant, dans le cas σ-fini, on utilise la formule de changement de variables des intégrales stables et l'on considère $r : E \to \mathbb{R}_+$ telle que $\hat{m}(dx) = \frac{1}{r(x)} m(dx)$ soit une mesure de probabilité.

On se donne α une fonction de classe C^1 définie sur U et à valeurs dans $(0, 2)$. Soit b une fonction de classe C^1 définie et bornée sur U, et $f(t, u, .)$ une famille de fonctions telles que, pour tout $(t, u) \in U^2$, $f(t, u, .) \in \mathcal{F}_{\alpha(u)}(E, \mathcal{E}, m)$.

On considère le champ aléatoire :

$$X(t, u) = b(u)(m(E))^{1/\alpha(u)} C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} f(t, u, V_i), \tag{I.19}$$

où $C_\eta = \left(\int_0^\infty x^{-\eta} \sin(x) dx\right)^{-1}$. Lorsque la fonction α est constante, (I.19) correspond donc à la représentation de Ferguson - Klass - LePage d'une variable stable. Dans le cas σ-fini, on considère le champ :

$$X(t, u) = b(u) C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} r(V_i)^{1/\alpha(u)} f(t, u, V_i). \tag{I.20}$$

Les processus multistables sont alors obtenus en prenant la diagonale du champ X, *i.e.*

$$Y(t) = X(t, t). \tag{I.21}$$

Le résultat principal du Chapitre III donne des critères sur les noyaux f pour obtenir des processus multistables localisables :

Théorème I.20. *On considère $(E, \mathcal{E}, m)$ un espace de mesure finie, avec $m \not\equiv 0$, et le champ aléatoire $X(., .)$ défini par (I.19). On suppose que $X(., u)$ est localisable au point u d'exposant $h \in (0, 1)$ et de forme locale $X'_u(., u)$. On suppose de plus que :*

- *(C1) La famille de fonctions $v \to f(t, v, x)$ est différentiable pour tout (v, t) dans un voisinage du point u et presque tout x dans E. La dérivée de f par rapport à v est notée f'_v.*
- *(C2) Il existe $\varepsilon > 0$ tel que :*

$$\sup_{t\in B(u,\varepsilon)} \int_E \sup_{w\in B(u,\varepsilon)} \left(|f(t, w, x)|^{\alpha(w)}\right) \hat{m}(dx) < \infty. \tag{I.22}$$

- *(C3) Il existe $\varepsilon > 0$ tel que :*

$$\sup_{t\in B(u,\varepsilon)} \int_E \sup_{w\in B(u,\varepsilon)} \left(|f'_v(t, w, x)|^{\alpha(w)}\right) \hat{m}(dx) < \infty. \tag{I.23}$$

- *(C4) Il existe $\varepsilon > 0$ tel que :*

$$\sup_{t\in B(u,\varepsilon)} \int_E \sup_{w\in B(u,\varepsilon)} \left[|f(t,w,x)\log|f(t,w,x)||^{\alpha(w)} \right] \hat{m}(dx) < \infty. \tag{I.24}$$

Alors $Y(t) \equiv X(t,t)$ est localisable au point u d'exposant h et de forme locale $Y'_u(t) = X'_u(t,u)$.

On donne également des versions multistables de processus stables classiques, comme par exemple le processus de Lévy multistable.

Théorème I.21 (Mouvement de Lévy symétrique multistable, cas compact)**.** *Soit $\alpha : [0,T] \to (1,2)$ et $b : [0,T] \to \mathbb{R}^+$ des fonctions de classe $\mathcal{C}^1$. Soit $(\Gamma_i)_{i\geq 1}$ une suite de temps d'arrivée d'un processus de Poisson d'intensité 1, $(\gamma_i)_{i\geq 1}$ de variables aléatoires indépendantes identiquement distribuées selon la loi $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$, et $(V_i)_{i\geq 1}$ un suite de variables aléatoires indépendantes et identiquement distribuées selon la loi uniforme sur $[0,T]$. On suppose que les trois suites $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, et $(\gamma_i)_{i\geq 1}$ sont indépendantes, et on définit*

$$Y(t) = b(t) C_{\alpha(t)}^{1/\alpha(t)} T^{1/\alpha(t)} \sum_{i=1}^{+\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} 1_{[0,t]}(V_i) \quad (t \in [0,T]). \tag{I.25}$$

Alors Y est $1/\alpha(u)$-localisable en tout point $u \in (0,T)$, de forme locale $Y'_u = b(u) L_{\alpha(u)}$.

Dans le chapitre IV, nous étudions la régularité locale des processus multistables, avec notamment l'étude de l'exposant de Hölder ponctuel. On considère les moments des processus multistables, et nous établissons ensuite un lien entre l'exposant de Hölder ponctuel et celui de localisabilité.

Théorème I.22. *Soit $t \in \mathbb{R}$ et U un intervalle ouvert de $\mathbb{R}$ avec $t \in U$. Soit $\eta \in (0,c)$. Alors, sous des hypothèses techniques sur f, quand ε tend vers 0,*

$$\mathsf{E}\left[|Y(t+\varepsilon) - Y(t)|^\eta\right] \sim \varepsilon^{\eta h(t)} \mathsf{E}\left[|Y'_t(1)|^\eta\right].$$

Théorème I.23 (Borne supérieure de l'exposant de Hölder)**.** *Soit $t \in U$. On suppose que Y est $h(t)$-localisable au point t. Sous des hypothèses techniques sur f, on a, presque sûrement :*

$$\mathcal{H}_t \leq h(t), \tag{I.26}$$

où $\mathcal{H}_t = \sup\{\gamma : \lim\limits_{r\to 0} \frac{|Y(t+r)-Y(t)|}{|r|^\gamma} = 0\}$.

Le cas particulier du processus de Lévy multistable est également étudié ici. Nous établissons dans ce cas que l'inégalité (I.26) est en fait une égalité.

Enfin le chapitre V, extrait de [31], aborde la question de l'estimation de la fonction de stabilité α et de la fonction de localisabilité h. Etant donnée une trajectoire observée, nous donnons deux estimateurs de ces fonctions, convergeant dans tous les espaces L^p.

Les observations de la trajectoire du processus multistable Y se font par pas de $\frac{1}{N}$. On définit la suite $(Y_{k,N})_{k\in\mathbb{Z},N\in\mathbb{N}}$ par

$$Y_{k,N} = Y(\frac{k+1}{N}) - Y(\frac{k}{N}).$$

Soit $t_0 \in \mathbb{R}$ fixé. On introduit alors un estimateur de $H(t_0)$ avec

$$\hat{H}_N(t_0) = -\frac{1}{n(N)\log N} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \log|Y_{k,N}|$$

où $(n(N))_{N\in\mathbb{N}}$ est une suite d'entiers pairs. On obtient alors la convergence de $\hat{H}_N$.

Théorème I.24. *Soit Y un processus multistable. On suppose que $\lim\limits_{N\to+\infty} \frac{N}{n(N)} = +\infty$.*

Alors, sous des hypothèses techniques sur f, pour tout $t_0 \in U$ et tout $r > 0$,

$$\lim_{N\to+\infty} \mathsf{E}\left|\hat{H}_N(t_0) - H(t_0)\right|^r = 0.$$

Pour l'estimation de la fonction α, on considère les moments empiriques $S_N(p)$ définis par

$$S_N(p) = \left(\frac{1}{n(N)} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} |Y_{k,N}|^p\right)^{\frac{1}{p}},$$

où $p_0 > 0$ et $\gamma \in (0,1)$.

On note

$$R_{\exp}(p) = \frac{S_N(p_0)}{S_N(p)} \text{ et } R_\alpha(p) = \frac{(\mathsf{E}|Z|^{p_0})^{1/p_0}}{(\mathsf{E}|Z|^p)^{1/p}} \mathbf{1}_{p<\alpha}$$

où $Z \sim S_\alpha(1,0,0)$.

On définit ensuite un estimateur de $\alpha(t_0)$ par

$$\hat{\alpha}_N(t_0) = \min\left(\arg\min_{\alpha\in[0,2]} \left(\int_{p_0}^2 |R_{\exp}(p) - R_\alpha(p)|^\gamma dp\right)^{1/\gamma}\right).$$

On obtient également la convergence de $\hat{\alpha}_N$.

Théorème I.25. *Soit Y un processus multistable. On suppose que :*

- $\lim\limits_{N\to+\infty} n(N) = +\infty$.
- $\lim\limits_{N\to+\infty} \frac{N}{n(N)} = +\infty$.
- *Le processus $X(.,t_0)$ est $H(t_0)$-auto-similaire à accroissements stationnaires et $H(t_0) < 1$.*

- *(C*) Il existe* $\epsilon_1 > 0$ *et* $j_0 \in \mathbb{N}$ *tels que pour tout* $j \geq j_0$,

$$\int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} m(dx) \leq (1-\epsilon_1)\|h_{0,t_0}\|_{\alpha(t_0)}^{\alpha(t_0)},$$

où $h_{j,u}(x) = f(j+1,u,x) - f(j,u,x)$.

- $\lim\limits_{j\to+\infty} \int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} m(dx) = 0.$

Alors, sous des hypothèses techniques sur f*, pour tout* $t_0 \in U$ *et* $r > 0$,

$$\lim_{N\to+\infty} \mathsf{E}\,|\hat{\alpha}_N(t_0) - \alpha(t_0)|^r = 0.$$

Nous illustrons les performances de ces estimateurs à partir de deux exemples, le Mouvement de Lévy Multistable et le Mouvement Linéaire Multifractionnaire Multistable.

Chapitre II

Moyennes mobiles symétriques stables et processus multistables localisables

Plan du chapitre

Abstract

We study a particular class of moving average processes which possess a property called *localisability*. This means that, at any given point, they admit a "tangent process", in a suitable sense. We give general conditions on the kernel g defining the moving average which ensures that the process is localisable and we characterize the nature of the associated tangent processes. Examples include the reverse Ornstein-Uhlenbeck process and the multistable reverse Ornstein-Uhlenbeck process. In the latter case, the tangent process is, at each time t, a Lévy stable motion with stability index possibly varying with t. We also consider the problem of path synthesis, for which we give both theoretical results and numerical simulations.

The remainder of this paper is organized as follows : in Section II.1, we give general conditions on the kernel g defining the moving average process to ensure (strong) localisability. Section II.2 specializes these conditions to cases where explicit forms for the tangent process may be given, and presents some examples. In Section II.3, we deal with multistable moving average processes, which generalize moving average stable processes by letting the stability index vary over time. Finally, Section II.4 considers numerical aspects : for applications, it is desirable to synthesize paths of these processes. Using the approach developed in [53], we first explain how to build traces of arbitrary moving average stable processes. In the case where the processes are localisable, we then give error bounds between the numerical and theoretical paths. Under mild additional assumptions, an 'optimal' choice of the parameters defining the synthesis method is derived. Finally, traces obtained from numerical experiments are displayed.

II.1 Localisability of stable moving average processes

We will be concerned with a special kind of stable processes that are stationary and may be expressed as moving average stochastic integrals in the following way :

$$Y(t) = \int g(t-x)M(dx) \quad (t \in \mathbb{R}), \tag{II.1}$$

where $g \in \mathcal{F}_\alpha$ is sometimes called the *kernel* of Y.

Such processes are considered in several areas (e.g. linear time-invariant systems) and it is of interest to know under what conditions they are localisable. A sufficient condition is provided by the following proposition.

Proposition II.1. Let $0 < \alpha \leq 2$ and let M be a symmetric α-stable measure on $\mathbb{R}$ with control measure Lebesgue measure $\mathcal{L}$. Let $g \in \mathcal{F}_\alpha$ and let Y be the moving average process

$$Y(t) = \int g(t-x)M(dx) \quad (t \in \mathbb{R}).$$

Suppose that there exist jointly measurable functions $h(t,.) \in \mathcal{F}_\alpha$ such that

$$\lim_{r\to 0^+} \int \left| \frac{g(r(t+z)) - g(rz)}{r^\gamma} - h(t,z) \right|^\alpha dz = 0 \tag{II.2}$$

for all $t \in \mathbb{R}$, where $\gamma + 1/\alpha > 0$. Then Y is $(\gamma + 1/\alpha)$-localisable with local form $Y'_u = \{\int h(t,z)M(dz) : t \in \mathbb{R}\}$ at all $u \in \mathbb{R}$.

Proof

Using stationarity followed by a change of variable $z = -x/r$ and the self-similarity of M,

$$\begin{aligned} Y(u+rt) - Y(u) &= Y(rt) - Y(0) \\ &= \int (g(rt-x) - g(-x))M(dx) \\ &= r^{1/\alpha} \int (g(r(t+z)) - g(rz))M(dz), \end{aligned}$$

where equalities are in finite dimensional distributions. Thus

$$\frac{Y(u+rt)-Y(u)}{r^{\gamma+1/\alpha}} - \int h(t,z)M(dz) = \int \left(\frac{g(r(t+z))-g(rz)}{r^{\gamma}} - h(t,z)\right) M(dz).$$

By [49, Proposition 3.5.1] and (II.2), $r^{-\gamma-1/\alpha}(Y(u+rt)-Y(u)) \to \int h(t,z)M(dz)$ in probability and thus in finite dimensional distributions ■

A particular instance of (II.1) is the reverse Ornstein-Uhlenbeck process, see [49, Section 3.6]. This process provides a straightforward application of Proposition II.1. One could also consider the Ornstein-Uhlenbeck process, which satisfies the conditions of Proposition II.1 too.

Proposition II.2 (Reverse Ornstein-Uhlenbeck process)**.** Let $\lambda > 0$ and $1 < \alpha \le 2$ and let M be an α-stable measure on $\mathbb{R}$ with control measure $\mathcal{L}$. The stationary process

$$Y(t) = \int_t^{\infty} \exp(-\lambda(x-t))M(dx) \quad (t \in \mathbb{R})$$

has a version in $D(\mathbb{R})$ that is $1/\alpha$-localisable at all $u \in \mathbb{R}$ with $Y'_u = L_\alpha$, where $L_\alpha(t) := \int_0^t M(dz)$ is α-stable Lévy motion.

Proof

The process Y is a moving average process that may be written in the form (II.1) with $g(x) = \exp(\lambda x)\mathbf{1}_{(-\infty,0]}(x)$. It is easily verified using the dominated convergence theorem that g satisfies (II.2) with $\gamma = 0$ and $h(t,z) = -\mathbf{1}_{(-t,0]}(z)$ for $t \ge 0$ and $h(t,z) = -\mathbf{1}_{(0,-t]}(z)$ for $t < 0$, so Proposition II.1 gives the conclusion with $Y'_u(t) = -M([-t,0]) = L_\alpha(t)$ for $t \ge 0$ and a similar formula for $t < 0$ ■

Proposition II.1 gives a condition on the kernel ensuring localisability. With an additional constraint we can get strong localisability. First we need the following proposition on continuity.

Proposition II.3. Let $0 < \alpha < 2$, $g \in \mathcal{F}_\alpha$ and let M be an α-stable symmetric random measure on $\mathbb{R}$ with control measure $\mathcal{L}$. Consider the moving average process defined by (II.1). Suppose that g satisfies, for all sufficiently small h,

$$\int |g(h-x)-g(-x)|^{\alpha}\, dx \le c|h|^{\lambda},$$

where $c > 0$ and $\lambda > 1$. Then Y has a continuous version which satisfies a θ-Hölder condition for all $\theta < (\lambda-1)/\alpha$.

Proof

By stationarity,

$$\begin{aligned} Y(t)-Y(t') &= Y(t-t')-Y(0) \\ &= \int \big(g(t-t'-x)-g(-x)\big)\, M(dx). \end{aligned}$$

So for $0 < p < \alpha$

$$\begin{aligned} \mathsf{E}|Y(t) - Y(t')|^p &\leq c_1 \left(\int |g(t-t'-x) - g(-x)|^\alpha \, dx \right)^{p/\alpha} \\ &\leq c_2 |t-t'|^{\lambda p/\alpha}. \end{aligned}$$

The result then follows from the Kolmogorov criterion by taking p arbitrarily close to α ■

Proposition II.4. With the same notation and assumptions as in Proposition II.1, suppose that in addition that g satisfies, for all sufficiently small h,

$$\int |g(h-x) - g(-x)|^\alpha \, dx \leq c|h|^{\alpha\gamma+1}, \tag{II.3}$$

where $c > 0$ and $\gamma > 0$. Then Y has a version in $C(\mathbb{R})$ that is $(\gamma + 1/\alpha)$-strongly localisable with $Y'_u = \{\int h(t,z)M(dz) : t \in \mathbb{R}\}$ at all $u \in \mathbb{R}$.

Proof

By Proposition II.3, $Y(t)$ has a continuous version and so $Z_r(t) := r^{-(\gamma+1/\alpha)}(Y(rt) - Y(0))$ also has a continuous version. Thus, for $0 < p < \alpha$, by stationarity and setting $h = r|t-t'|$ sufficiently small,

$$\begin{aligned} \mathsf{E}|Z_r(t) - Z_r(t')|^p &= \mathsf{E}|Z_r(t-t') - Z_r(0)|^p \\ &= c_1 \left(\int \left| \frac{g(r(t-t')-x) - g(-x)}{r^{\gamma+1/\alpha}} \right|^\alpha dx \right)^{p/\alpha} \\ &= c_1 \left(\int \frac{|g(h-x) - g(-x)|^\alpha}{h^{\gamma\alpha+1}} dx \right)^{p/\alpha} |t-t'|^{(\gamma\alpha+1)p/\alpha} \\ &\leq c_2 |t-t'|^{(\gamma\alpha+1)p/\alpha}, \end{aligned}$$

provided $|t-t'|$ is sufficiently small, using (II.3) in the last step. We may choose p sufficiently close to α so that $(\gamma\alpha+1)p/\alpha > 1$. By a corollary to Kolmogorov's criterion (see e.g. [47, Theorem 85.5]) the measures on $C(\mathbb{R})$ underlying the processes Z_r are conditionally compact. Thus convergence in finite dimensional distributions of Z_r to Y'_u as $r \searrow 0$ implies the convergence in distribution (with Y'_u necessarily having a continuous version). Together with localisability which follows from Proposition II.1 this gives strong localisability ■

Note that the reverse Ornstein-Uhlenbeck process is a stationary Markov process which has a version in $D(\mathbb{R})$ see [50, Remark 17.3]. It also satisfies (II.3) for $\alpha \geq 1$ with $\gamma = 0$. However, we cannot deduce that it is strongly localisable since Proposition II.4 is only valid for $\gamma > 0$. The case $\gamma = 0$ would be interesting to deal with, but is much harder and would require different techniques.

II.2 Sufficient conditions for localisability and examples

For the reverse Ornstein-Uhlenbeck process, it was straightforward to check the conditions of Proposition II.1. In general, however, it is not easy to guess which kind of functions g in $\mathcal{F}_\alpha$ will satisfy (II.2). In this section we will find simple practical conditions ensuring this.

Recall that the following process is called *linear fractional α-stable motion* :

$$L_{\alpha,H,b^+,b^-}(t) = \int_{-\infty}^{\infty} f_{\alpha,H}(b^+, b^-, t, x) M(dx),$$

where $t \in \mathbb{R}$, $b^+, b^- \in \mathbb{R}$, and

$$\begin{aligned} f_{\alpha,H}(b^+, b^-, t, x) = b^+ &\left((t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}\right) \\ &+ b^- \left((t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha}\right), \end{aligned} \tag{II.4}$$

where M is a symmetric α-stable random measure ($0 < \alpha < 2$) with control measure Lebesgue measure. Being sssi, L_{α,H,b^+,b^-} is localisable. In addition, it is strongly localisable when $H > 1/\alpha$, since its paths then belong to $C(\mathbb{R})$.

Recall also that the process

$$L_\alpha(t) = \int_0^t M(dx) \tag{II.5}$$

is *α-stable Lévy motion* and the process

$$Z_\alpha(t) = \int_{-\infty}^{+\infty} (\ln|t-x| - \ln|x|) M(dx) \tag{II.6}$$

is called *log-fractional stable motion.*

We are now ready to describe easy-to-check conditions that ensure that Propositions II.1 and II.4 apply.

Proposition II.5. Let $0 < \alpha \leq 2$, $g \in \mathcal{F}_\alpha$ and M be an α-stable symmetric random measure on $\mathbb{R}$ with control measure $\mathcal{L}$. Let Y be the moving average process

$$Y(t) = \int g(t-x) M(dx) \qquad (t \in \mathbb{R}).$$

If there exist $c_0^+, c_0^-, \gamma, a, c, \eta \in \mathbb{R}$ with $c > 0$, $\eta > 0$ and $0 < \gamma + 1/\alpha < a \leq 1$ such that

$$\frac{g(r)}{r^\gamma} \to c_0^+ \text{ and } \frac{g(-r)}{r^\gamma} \to c_0^- \tag{II.7}$$

as $r \searrow 0$ and

$$|g(u+h) - g(u)| \leq c|h|^a |u|^{\gamma - a} \quad (u \in \mathbb{R}, |h| < \eta), \tag{II.8}$$

then Y is $(\gamma + 1/\alpha)$-localisable at all $u \in \mathbb{R}$ with local form

$$(a) \quad Y_u' = L_{\alpha,\gamma+1/\alpha,c_0^+,c_0^-} \quad \text{if } \gamma \neq 0,$$

$$(b) \quad Y_u' = (c_0^+ - c_0^-) L_\alpha \quad \text{if } \gamma = 0.$$

If, in addition, $\gamma > 0$ and $0 < \alpha < 2$ then Y has a version in $C(\mathbb{R})$ and is strongly localisable.

Note that condition (II.8) on the increments of g may be interpreted as a 2-microlocal condition, namely that g belongs to the global 2-microlocal space $C_0^{\gamma,a-\gamma}$, see [36]. Remark also that, in order for this condition to be satisfied by non-trivial functions g, one needs $a \leq 1$, which in turns implies that $\gamma \leq 1 - 1/\alpha$ and $a - \gamma \in (1/\alpha, 1-\gamma]$. Finally note that condition (II.7) induces a special behaviour of the kernel near the origin. By a stationary argument, one could equivalently give a sufficient condition for localisability which involves a similar condition near any other point.

Proof

(a) We have

$$\begin{aligned}\frac{g(r(t+z))-g(rz)}{r^\gamma} = \frac{g(r|t+z|)}{(r|t+z|)^\gamma}|t+z|^\gamma \mathbf{1}_{\{t+z\geq 0\}} + \frac{g(-r|t+z|)}{(r|t+z|)^\gamma}|t+z|^\gamma \mathbf{1}_{\{t+z<0\}} \\ - \frac{g(r|z|)}{(r|z|)^\gamma}|z|^\gamma \mathbf{1}_{\{z\geq 0\}} - \frac{g(-r|z|)}{(r|z|)^\gamma}|z|^\gamma \mathbf{1}_{\{z<0\}}.\end{aligned}$$

As $r \to 0$,

$$\begin{aligned}\frac{g(r(t+z))-g(rz)}{r^\gamma} &\to c_0^+|t+z|^\gamma \mathbf{1}_{\{t+z\geq 0\}} + c_0^-|t+z|^\gamma \mathbf{1}_{\{t+z<0\}} - c_0^+|z|^\gamma \mathbf{1}_{\{z\geq 0\}} - c_0^-|z|^\gamma \mathbf{1}_{\{z<0\}} \\ &= c_0^+(t+z)_+^\gamma - c_0^+(z)_+^\gamma + c_0^-(t+z)_-^\gamma - c_0^-(z)_-^\gamma \qquad \text{(II.9)} \\ &= f_{\alpha,\gamma+1/\alpha}(c_0^+, c_0^-, t, -z).\end{aligned}$$

To get convergence in L^α we use the dominated convergence theorem. Fix $\epsilon > 0$ and $m > 0$ such that for all $0 < u < \epsilon$,

$$\left|\frac{g(u)}{u^\gamma} - c_0^+\right| \leq m \text{ and } \left|\frac{g(-u)}{u^\gamma} - c_0^-\right| \leq m.$$

For fixed t write $f_r(z) = r^{-\gamma}(g(r(t+z)) - g(rz))$. If $t = 0$,

$$f_r(z) = f_{\alpha,\gamma+1/\alpha}(c_0^+, c_0^-, t, -z) = 0,$$

thus $f_r(.) \to f_{\alpha,\gamma+1/\alpha}(c_0^+, c_0^-, t, -.)$ which belongs to L^α. Assume now $t \in \mathbb{R}^*$. There is a constant m_1 such that $|f_r(z)|^\alpha \leq m_1(1+|z|^\gamma+|t+z|^\gamma)^\alpha$ for all $|r| \leq \frac{\epsilon}{1+|t|}$ and $|z| \leq 1$. From (II.8)

$$|f_r(z)|^\alpha \leq \left(\frac{|rt|^a|rz|^{\gamma-a}}{|r|^\gamma}\right)^\alpha \leq |t|^{a\alpha}|z|^{(\gamma-a)\alpha}$$

for $|r| < \eta/|t|$, so, as $(\gamma - a)\alpha < -1$ and $\gamma\alpha > -1$,

$$\int_{|z|\leq 1} m_1(1+|z|^\gamma+|t+z|^\gamma)^\alpha dz + \int_{|z|>1} |z|^{(\gamma-a)\alpha} dz < \infty.$$

Since also $f_{\alpha,\gamma+1/\alpha}(c_0^+, c_0^-, t, -z) \in L^\alpha$, the dominated convergence theorem implies that $f_r(z) \to f_{\alpha,\gamma+1/\alpha}(c_0^+, c_0^-, t, -z)$ in L^α. The conclusion in case (a) follows from Proposition II.1, (II.4), and noting that M is a symmetric α-stable measure.

(b) In this case the limit (II.9) is

$$\frac{g(r(t+z))-g(rz)}{r^\gamma} \to \begin{cases} (c_0^+-c_0^-)\mathbf{1}_{[0,t]}(-z) & \text{if} \quad t\geq 0 \\ -(c_0^+-c_0^-)\mathbf{1}_{[t,0]}(-z) & \text{if} \quad t<0 \end{cases}.$$

Dominated convergence follows in the same way as in case (a) so the conclusion follows from Proposition II.1 and (II.5).

Moving to strong localisability, for h small enough,

$$\int_{|x|\leq 3|h|}|g(h-x)-g(-x)|^\alpha dx \leq c_1 \int_{|x|\leq 3|h|}|h|^{\alpha\gamma}dx \leq c_2|h|^{\alpha\gamma+1},$$

and

$$\begin{aligned} \int_{|x|\geq 3|h|}|g(h-x)-g(-x)|^\alpha dx &\leq c_1|h|^{a\alpha}\int_{3|h|}^\infty |x|^{(\gamma-a)\alpha}dx \\ &\leq c_2|h|^{a\alpha}|h|^{1+(\gamma-a)\alpha} \\ &= c_2|h|^{\alpha\gamma+1} \end{aligned}$$

and the conclusion follows from Propositions II.3 and II.4 ■

We now give an alternative condition for localisability in terms of Fourier transforms. Note that the Fourier transform $\widehat{f}_{\alpha,H}(b^+,b^-,t,\xi)$ of $f_{\alpha,H}(b^+,b^-,t,.)$ is given by

$$\begin{aligned} \widehat{f}_{\alpha,H}(b^+,b^-,t,\xi) =& \Gamma(H+1-1/\alpha)\frac{e^{-i\xi t}-1}{|\xi|^{H+1-1/\alpha}} \\ &\times\left[b^+\exp\left(\frac{i\pi}{2}\mathrm{sgn}(\xi)(H+1-1/\alpha)\right)+b^-\exp\left(-\frac{i\pi}{2}\mathrm{sgn}(\xi)(H+1-1/\alpha)\right)\right]. \end{aligned}$$

Proposition II.6. Let $1\leq\alpha\leq 2$, and Y be defined by (II.1). If there exist $l=l_1+il_2\in\mathbb{C}^*$, $\gamma\in(-\frac{1}{\alpha},1-\frac{1}{\alpha})$, $a\in(0,1-(\gamma+\frac{1}{\alpha}))$ and $K\in L^p(\mathbb{R})$ with $p\in[1,1/(\gamma+\frac{1}{\alpha}+a))$, such that for almost all $\xi>0$,

$$\xi^{\gamma+1}\widehat{g}(\xi) = l+\frac{1}{\xi^a}\widehat{K}(\xi), \tag{II.10}$$

then Y is $(\gamma+1/\alpha)$-localisable at all $u\in\mathbb{R}$ with local form

$$(a) \quad Y'_u = L_{\alpha,\gamma+1/\alpha,b^+,b^-} \quad \text{if } \gamma\neq 0,$$

$$(b) \quad Y'_u = \tfrac{1}{\pi}l_1 Z_\alpha + l_2 L_\alpha \quad \text{if } \gamma = 0,$$

where

$$\begin{aligned} b^+ &= \frac{1}{2\Gamma(\gamma+1)}\left(\frac{l_1}{\cos(\pi(\gamma+1)/2)}-\frac{l_2}{\sin(\pi(\gamma+1)/2)}\right), \\ b^- &= \frac{1}{2\Gamma(\gamma+1)}\left(\frac{l_1}{\cos(\pi(\gamma+1)/2)}+\frac{l_2}{\sin(\pi(\gamma+1)/2)}\right). \end{aligned}$$

Proof (a)
First note that, with b^+ and b^- as above, we have, for $z \neq 0$,

$$\widehat{f}_{\alpha,\gamma+1/\alpha}(b^+, b^-, t, \xi) = \frac{e^{-i\xi t} - 1}{|\xi|^{\gamma+1}}(\bar{l}\mathbf{1}_{\xi>0} + l\mathbf{1}_{\xi<0}).$$

Set $f_r(z) = r^{-\gamma}(g(r(t+z)) - g(rz))$. Then $f_r \in \mathcal{F}_\alpha$ and

$$\widehat{f_r}(\xi) = \frac{e^{i\xi t} - 1}{r^{\gamma+1}}\widehat{g}(\frac{\xi}{r}).$$

With α' such that $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ we have $\widehat{f_r} \in \mathcal{F}_{\alpha'}$ and $\widehat{f}_{\alpha,\gamma+1/\alpha}(b^+, b^-, t, \xi) \in \mathcal{F}_{\alpha'}$. We now show that $\|f_r(.) - f_{\alpha,\gamma+1/\alpha}(b^+, b^-, t, -\cdot)\|_\alpha \to 0$ when $r \to 0$. Note that (II.10) implies that for $\xi < 0$

$$|\xi|^{\gamma+1}\widehat{g}(\xi) = \bar{l} + \frac{1}{|\xi|^a}\widehat{K}(\xi).$$

Writing $\widehat{f}(\xi) = \widehat{f}_{\alpha,\gamma+1/\alpha}(b^+, b^-, t, -\xi)$, for almost all $\xi \in \mathbb{R}$

$$\begin{aligned}
\widehat{f_r}(\xi) - \widehat{f}(\xi) &= \frac{(e^{i\xi t} - 1)}{|\xi|^{\gamma+1}}\Big(\Big(\frac{|\xi|^{\gamma+1}}{r^{\gamma+1}}\widehat{g}(\frac{\xi}{r}) - l\Big)\mathbf{1}_{\xi>0} + \Big(\frac{|\xi|^{\gamma+1}}{r^{\gamma+1}}\widehat{g}(\frac{\xi}{r}) - \bar{l}\Big)\mathbf{1}_{\xi<0}\Big) \\
&= \frac{(e^{i\xi t} - 1)}{|\xi|^{\gamma+1}}\frac{r^a}{|\xi|^a}K(\frac{\xi}{r}) \\
&= r^a\frac{(e^{i\xi t} - 1)}{|\xi|^{\gamma+1+a}}K(\frac{\xi}{r}).
\end{aligned}$$

Let $H_r(u) = K(ru)$. Then $\widehat{H}_r(\xi) = \frac{1}{r}\widehat{K}(\frac{\xi}{r})$ and we may write for $a + \gamma \neq 0$

$$\widehat{f_r}(\xi) - \widehat{f}(\xi) = r^{a+1}\widehat{f}_{\alpha,\gamma+1/\alpha+a}(b, b, t, -\xi)\widehat{H_r}(\xi), \tag{II.11}$$

where $b = 1/(2\Gamma(\gamma + a + 1)\cos(\pi(\gamma + a + 1)/2))$.

It is easy to verify that $f_{\alpha,\gamma+a+1/\alpha}(b^+, b^-, t, -\cdot) \in L_\beta$ for all $\beta > 1/(1 - \gamma - a)$. By the conditions on α and p, there exists such a β which also satisfies $\frac{1}{\alpha} + 1 = \frac{1}{\beta} + \frac{1}{p}$ and in particular, $\frac{1}{p} + \frac{1}{\beta} > 1$. Consequently we may take the inverse Fourier transform of (II.11) see, for example, [56, Theorem 78] to get :

$$f_r(z) - f(z) = r^{a+1}f_{\alpha,\gamma+1/\alpha+a}(b^+, b^-, t, -.) * H_r(z)$$

where $*$ denotes convolution. As $\frac{1}{\alpha} + 1 = \frac{1}{\beta} + \frac{1}{p}$, the Hausdorff-Young inequality yields

$$\begin{aligned}
\|f_r - f_{\alpha,\gamma+1/\alpha}(b^+, b^-, t, -\cdot)\|_\alpha &\leq r^{a+1}\|f_{\alpha,\gamma+1/\alpha+a}\|_\beta\|H_r\|_p \\
&\leq r^{a+1-\frac{1}{p}}\|f_{\alpha,\gamma+1/\alpha+a}\|_\beta\|K\|_p.
\end{aligned}$$

We conclude that $f_r \to f_{\alpha,\gamma+1/\alpha}(b^+, b^-, t, -\cdot)$ in L^α. The result follows from Proposition II.1. The case $a + \gamma = 0$ is dealt with in a similar way.

(b) Let z_t and l_t be defined by

$$l_t(x) = \begin{cases} \mathbf{1}_{]0,t[}(x) & \text{if } t \geq 0 \\ -\mathbf{1}_{]t,0[}(x) & \text{if } t < 0 \end{cases}$$

and

$$z_t(x) = \ln|t-x| - \ln|x|.$$

A straightforward computation shows that

$$z_t(x) = \operatorname{sgn}(-t) \lim_{\varepsilon \to 0} \int_{|s| \geq \varepsilon} \frac{1}{s} \mathbf{1}_{[\min(x-t,x),\max(x-t,x)]}(s) ds,$$

so that, in the space of distributions we get

$$z_t = -\mathrm{PV}(1/\cdot) * l_t$$

where PV denotes the Cauchy principal value. Thus

$$\begin{aligned} \widehat{z_t}(\xi) &= -\widehat{\mathrm{PV}(1/\cdot)}(\xi)\widehat{l_t}(\xi) \\ &= -(-i\pi \operatorname{sgn}(\xi))\big(-\frac{1}{i\xi}(e^{-i\xi t}-1)\big) \\ &= -\pi \frac{e^{-i\xi t}-1}{|\xi|}. \end{aligned}$$

With $f(z) = -\frac{1}{\pi} l_1 z_t(-z) - l_2 l_t(-z)$, we obtain

$$\widehat{f}(\xi) = \frac{e^{i\xi t}-1}{|\xi|}(l\mathbf{1}_{\xi>0} + \bar{l}\mathbf{1}_{\xi<0}).$$

As in (a) we conclude that $f_r \to f$ in L^α. Proposition II.1 implies that Y is $(\gamma + 1/\alpha)$-localisable at all $u \in \mathbb{R}$ with local form $Y'_u = \frac{1}{\pi} l_1 Z_\alpha + l_2 L_\alpha$, since M is symmetric ■

We give examples to illustrate Propositions II.5 and II.6.

Example II.7. Let $\frac{6}{5} < \alpha \leq 2$ and let M be an α-stable symmetric random measure on $\mathbb{R}$ with control measure $\mathcal{L}$. Let

$$g(x) = \begin{cases} 0 & (x \leq 0) \\ x^{1/6} & (0 < x \leq 1) \\ x^{-5/6} & (x \geq 1) \end{cases} .$$

The stationary process defined by

$$Y(t) = \int g(t-x) M(dx) \quad (t \in \mathbb{R})$$

is $(1/6 + 1/\alpha)$-strongly localisable at all $u \in \mathbb{R}$ with local form $Y'_u = L_{\alpha,1/6+1/\alpha,1,0}$.

Proof
We apply Proposition II.5 case (a) with $\alpha \in (\frac{5}{6}, 2]$. The function g satisfies the assumptions with $\gamma = \frac{1}{6}$, $c_0^+ = 1$, $c_0^- = 0$ and $a = 1$ ∎

To verify condition (II.10) of Proposition II.6, one needs to check that $g \in L^\alpha(\mathbb{R})$ and also that $\xi^{a+\gamma+1}\widehat{g}(\xi) - l\xi^a$ is the Fourier transform of a function in $L^p(\mathbb{R})$ for some a, γ, p in the admissible ranges. For this purpose, one may for instance apply classical theorems such as in [56, Theorems 82-84]. We give below an example that uses a direct approach.

Example II.8. For $1 \leq \alpha < 2$ let M be an α-stable symmetric random measure on $\mathbb{R}$ with control measure $\mathcal{L}$. Let g be defined by its Fourier transform

$$\widehat{g}(\xi) = \begin{cases} 0 & (|\xi| \leq 1) \\ |\xi|^{-\gamma-1} & (|\xi| > 1) \end{cases}$$

where $\gamma \in (-\frac{1}{\alpha}, \frac{1}{2} - \frac{1}{\alpha}) \subseteq (-1, 0)$. Then $g \in L^\alpha(\mathbb{R})$ and the moving average process

$$Y(t) = \int g(t-x)M(dx) \quad (t \in \mathbb{R})$$

is well-defined and $(\gamma + 1/\alpha)$-localisable at all $u \in \mathbb{R}$, with local form $Y'_u = L_{\alpha,\gamma+1/\alpha,b,b}$, where $b = -1/(2\Gamma(\gamma+1)\cos(\pi(\gamma+1/2)))$.

Proof
Taking $\widehat{K}(\xi) = |\xi|^{1/2}\mathbf{1}_{[-1,1]}(\xi)$ with $l = -1$ and $a = \frac{1}{2}$ in (II.10) gives g. To check that $K \in L^p(\mathbb{R})$ for all $p > 1$, note that K is continuous (in fact C^∞) and that $|K(x)| \leq C|x|^{-1}$ for all x. Then $Y(t)$ will be well-defined if g is in $L^\alpha(\mathbb{R})$. To verify this, one computes the inverse Fourier transform of $\widehat{g}$, to get $g(x) = 2(\gamma+1)|x|^\gamma \int_{|x|}^\infty |v|^{-\gamma-2}\sin v dv - 2x^{-1}\sin x$. By Proposition II.6(a), Y is α-localisable at all $u \in \mathbb{R}$ with the local form as stated ∎

The approach of this example may be used for general classes of functions g.

II.3 Multistable moving average processes

In [16], localisability is used to define *multistable processes*, that is processes which at each point $t \in \mathbb{R}$ have an $\alpha(t)$-stable random process as their local form, where $\alpha(t)$ is a sufficiently smooth function ranging in $(0, 2)$. Thus such processes "look locally like" a stable process at each t but with differing stability indices as time evolves.

Before we recall how this was done in [16], we note briefly that "stable-like" processes have been defined and studied in [39]. These stable-like processes are Markov jump processes, and are, in a sense, "localisable", but with localisability defined by the requirement that they are solutions of an order $\alpha(x)$ fractional stochastic differential equations. See Theorem 2.1 in [39], which shows that the local form of sample paths is considered rather than of the limiting process. Another essential difference is that stable-like processes are Markov, whereas, in general, multistable ones, as defined below, are not. In fact, formula (II.17), where a Poisson process element Y is independent of t but is raised to a power that involves t means that our processes are "far" from Markov.

We now come back to our multistable processes. One route to defining such processes is to rewrite stable integrals as countable sums over Poisson processes. We recall briefly how this can be done, see [16] for fuller details. Let $(E, \mathcal{E}, m)$ be a σ-finite measure space and let Π be a Poisson process on $E \times \mathbb{R}$ with mean measure $m \times \mathcal{L}$. Thus Π is a random countable subset of $E \times \mathbb{R}$ such that, writing $N(A)$ for the number of points in a measurable $A \subset E \times \mathbb{R}$, the random variable $N(A)$ has a Poisson distribution of mean $(m \times \mathcal{L})(A)$ with $N(A_1), \dots, N(A_n)$ independent for disjoint $A_1, \dots, A_n \subset E \times \mathbb{R}$, see [26]. In the case of constant α, with M a symmetric α-stable random measure on E with control measure m, one has, for $f \in \mathcal{F}_\alpha$ ([49, Section 3.12]),

$$\int f(x) M(dx) = c(\alpha) \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} f(\mathsf{X}) \mathsf{Y}^{<-1/\alpha>} \quad (0 < \alpha < 2), \tag{II.12}$$

where

$$c(\alpha) = \left(2\alpha^{-1} \Gamma(1-\alpha) \cos(\tfrac{1}{2}\pi\alpha)\right)^{-1/\alpha}, \tag{II.13}$$

and $a^{<b>} = \mathrm{sign}(a)|a|^b$.

Now define the random field

$$X(t, v) = \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} f(t, v, \mathsf{X}) \mathsf{Y}^{<-1/\alpha(v)>}. \tag{II.14}$$

Under certain conditions the "diagonal" process $X(t,t)$ gives rise to a multistable process with varying α of the form

$$Y(t) \equiv X(t,t) = \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} f(t, t, \mathsf{X}) \mathsf{Y}^{<-1/\alpha(t)>}. \tag{II.15}$$

Theorem 5.2 of [16] gives conditions on f that ensure that Y is localisable (or strongly localisable) with $Y'_u = X'_u(\cdot, u)$ at a given u, provided $X(\cdot, u)$ is itself localisable (resp. strongly localisable) at u. These conditions simplify very considerably in the moving average case, taking $E = \mathbb{R}$ and $m = \mathcal{L}$ with $f(t, v, x) = g(x - t)$. Our next theorem restates [16, Theorem 5.2] in this specific situation.

We need first to define a quasinorm on certain spaces of measurable functions on E. For $0 < a \le b < 2$ let

$$\mathcal{F}_{a,b} \equiv \mathcal{F}_{a,b}(E, \mathcal{E}, m) = \{f : f \text{ is } m\text{-measurable with } \|f\|_{a,b} < \infty\}$$

where

$$\|f\|_{a,b} = \left(\int_E |f(x)|^a m(dx)\right)^{1/a} + \left(\int_E |f(x)|^b m(dx)\right)^{1/b}. \tag{II.16}$$

Theorem II.9 (Multistable moving average processes). *Let U be a closed interval with u an interior point. Let $\alpha : U \to (a, b) \subset (0, 2)$ satisfy*

$$|\alpha(v) - \alpha(u)| \leq k_1 |v - u|^\eta \quad (v \in U)$$

where $0 < \eta \leq 1$. Let $g \in \mathcal{F}_{a,b}$, and define

$$Y(t) = \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} g(\mathsf{X} - t)\mathsf{Y}^{<-1/\alpha(t)>} \quad (t \in \mathbb{R}). \tag{II.17}$$

Assume that g satisfies, for all $t \in U$,

$$\lim_{r \to 0} \int \left| \frac{g(r(t+z)) - g(rz)}{r^\gamma} - h(t, z) \right|^{\alpha(u)} dz = 0 \tag{II.18}$$

for jointly measurable functions $h(t, \cdot) \in \mathcal{F}_{\alpha(u)}$, where $0 < \gamma + 1/\alpha(u) < \eta \leq 1$. Then Y is $(\gamma + 1/\alpha(u))$-localisable at u with local form $Y'_u = \{\int h(t, z) M_{\alpha(u)}(dz) : t \in \mathbb{R}\}$, where $M_{\alpha(u)}$ is the symmetric $\alpha(u)$-stable measure with control measure $\mathcal{L}$ and skewness 0.

Suppose further that $\gamma > 0$ and for h sufficiently small

$$\|g(h - x) - g(-x)\|_\alpha \leq c|h|^{\gamma + 1/\alpha(u)}.$$

Then Y has a continuous version and is strongly $(\gamma + 1/\alpha(u))$-localisable at u with local form $Y'_u = \{\int h(t, z) M_{\alpha(u)}(dz) : t \in \mathbb{R}\}$ under either of the following additional conditions :
(i) $0 < \alpha(u) < 1$ and g is bounded
(ii) $1 < \alpha(u) < 2$ and α is continuously differentiable on U with

$$|\alpha'(v) - \alpha'(w)| \leq k_1 |v - w|^\eta \quad (v, w \in U),$$

where $1/\alpha(u) < \eta \leq 1$.

Proof
Taking

$$X(t, v) = \sum_{(\mathsf{X},\mathsf{Y}) \in \Pi} g(\mathsf{X} - t)\mathsf{Y}^{<-1/\alpha(v)>} \quad (t, v \in \mathbb{R}). \tag{II.19}$$

this theorem is essentially a restatement of [16, Theorem 5.2] in the special case of $E = \mathbb{R}$ and $m = \mathcal{L}$ with $f(t, v, x) = g(x - t)$ in (II.14). Since $f(t, v, x)$ no longer depends on v most of the conditions in [16, Theorem 5.2] are trivially satisfied and we conclude that $Y'_u = X'_u(\cdot, u)$, noting that $X(\cdot, u)$ is $(\gamma + 1/\alpha(u))$-localisable (or strongly localisable) with the local form given by Propositions II.1 or II.4 ■

It is curious that neither cases (i) or (ii) address localisability if $\alpha(u) = 1$. This goes back to the proof of [3, Theorem 5.2] where different approaches are used in the two cases. For $\alpha(u) < 1$ the proof uses that the sum (4.8) is absolutely convergent almost surely, whereas for $\alpha(u) > 1$ we need to find a number p such that $0 < p < \alpha(t)$ for t near u with $\eta p > 1$ to enable us to apply Kolmogorov's criterion to certain increments.

Corollary II.10. Let U, α and g be as in Theorem II.9. Then the same conclusion holds if $Y(t)$ in (II.17) is replaced by $Y(t) = a(t)\sum_{(X,Y)\in\Pi} g(\mathsf{X}-t)\mathsf{Y}^{<-1/\alpha(t)>} \quad (t \in \mathbb{R})$, where a is a non-zero function of Hölder exponent $\eta > h$.

Proof
This follows easily in just the same way as Proposition 2.2 of [16] ∎

We may apply this theorem to get a multistable version of the reverse Ornstein-Uhlenbeck process considered in Section II.1 :

Proposition II.11 (Multistable reverse Ornstein-Uhlenbeck process)**.** Let $\lambda > 0$ and $\alpha : \mathbb{R} \to (1, 2)$ be continuously differentiable. Let

$$Y(t) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi, \mathsf{X}\geq t} \exp(-\lambda(\mathsf{X}-t))\mathsf{Y}^{<-1/\alpha(t)>} \quad (t \in \mathbb{R}).$$

Then Y is $1/\alpha(u)$-localisable at all $u \in \mathbb{R}$ with $Y'_u = c(\alpha(u))^{-1} L_{\alpha(u)}$, where L_α is α-stable Lévy motion.

Proof
Taking $g(x) = \mathbf{1}_{[0,\infty)}(x)\exp(-\lambda x)$ and $h(t,z) = -\mathbf{1}_{[-t,0]}(z)$ for $t \geq 0$ (and a similar formula for $t < 0$) with $\gamma = 0$, localisability follows from Theorem II.9 with the limit (II.18) being checked just as in Proposition II.2 ∎

Theorem II.9 applies in particular to functions g satisfying the conditions of Proposition II.5. Thus, for instance, the moving averages of Examples II.7 and II.8 admit multistable versions. The process of Example II.7 is strongly $\gamma + 1/\alpha(u)$ localisable at u whenever α verifies condition *(ii)*.

II.4 Path synthesis and numerical experiments

We address here the issue of path simulation. In the previous sections, we have considered two kinds of stochastic processes : moving average stable ones, that are stationary, and their multistable versions, which typically are not, nor have stationary increments. Our simulation method for the moving average stable processes is based on that presented in [53]. There, the authors propose an efficient algorithm for synthesizing paths of linear fractional stable motion. In fact, this algorithm really builds traces of the increments of linear fractional stable motion. These increments form a stationary process, an essential feature for the algorithm to work. It is straightforward to modify it to synthesize any stationary stable process which possesses an integral representation. In addition, we are able to obtain bounds on the approximation error measured in the α-norm, and thus on the r^{th} moments for $r < \alpha$, as shown below.

For non (increment) stationary processes, like multistable processes, a possibility would be to use the general method proposed in [8]. It allows to synthesize (fractional) fields defined by integration of a deterministic kernel with respect to a random infinitely divisible measure. When the control measure is finite, the idea is to approximate the integral with a generalized shot noise series. In this situation, a bound on the L^r norm of the error is obtained for appropriate r. In the case of infinite control measure, one needs to deal with the points "far from the origin" through a normal approximation. This second approximation maybe controlled through Berry-Esseen bounds which lead to a convergence in law. Thus the overall error when the control measure is infinite may only be assessed in law, and not in the stronger L^r norm.

Although the method of [8] may be used for the synthesis of multistable processes, we will rather take advantage here of the particular structure of our processes : being localisable, they are by definition tangent, at each point, to a stable process. Thus we may simulate them by "gluing" together in a appropriate way paths of their tangent processes, which are themselves synthesized through the simpler procedure of [53].

We briefly present in the next subsection the main ingredients of the method. We then give bounds estimating the errors entailed by the numeric approximation, in the case where the process is localisable. Finally, we display graphs of localisable moving average processes obtained with this synthesis scheme.

II.4.1 Simulation of stable moving averages

Let $Y = \{Y(t), t \in \mathbb{R}\}$ be the process defined by (II.1). To synthesize a path $Y(k), k = 1, ..., N, N \in \mathbb{N}$, of Y, the usual (Euler) method consists in approximating the integral by a Riemann sum. Two parameters tune the precision of the method : the discretization step ω and the cut-off value for the integral Ω. The idea in [53] is to use the fast Fourier transform for an efficient computation of the Riemann sum. More precisely, let

$$Y(k) = \int_{\mathbb{R}} g(k-s)dM(s) = -\int_{\mathbb{R}} g(s)dM(k-s).$$

Let $\omega, \Omega \in \mathbb{N}$ and

$$Y_{\omega,\Omega}(k) = \sum_{j=-\omega\Omega+1}^{0} g(\frac{j-1}{\omega})Z_{\alpha,\omega}(\omega k - j) + \sum_{j=1}^{\omega\Omega} g(\frac{j}{\omega})Z_{\alpha,\omega}(\omega k - j), \tag{II.20}$$

where $Z_{\alpha,\omega}(j) = M(\frac{j+1}{\omega}) - M(\frac{j}{\omega})$ are i.i.d. α-stable symmetric random variables. Let $Z_\alpha(j)$ denote a sequence of normalised i.i.d α-stable symmetric random variables. Then one has the equality in law : $\{Z_{\alpha,\omega}(j), j \in \mathbb{Z}\} = \{\omega^{-1/\alpha} Z_\alpha(j), j \in \mathbb{Z}\}$. One may thus write :

$$Y_{\omega,\Omega}(k) = \sum_{j=1}^{2\omega\Omega} a_\omega(j) Z_\alpha(\omega(k+\Omega) - j),$$

where

$$a_\omega(j) = \begin{cases} \omega^{-1/\alpha} g(\frac{j-1}{\omega} - \Omega) & \text{for } j \in \{1, ..., \omega\Omega\} \\ \omega^{-1/\alpha} g(\frac{j}{\omega} - \Omega) & \text{for } j \in \{\omega\Omega + 1, ..., 2\omega\Omega\}. \end{cases}$$

For $n \in \mathbb{Z}$, let

$$W(n) = \sum_{j=1}^{2\omega\Omega} a_\omega(j) Z_\alpha(n-j).$$

Then $\{Y_{\omega,\Omega}(k), k = 1, ..., N\}$ has the same law as $\{W(\omega(k+\Omega)), k = 1, ..., N\}$. But W is the convolution product of the sequences a_ω and Z_α. As such, it may be be efficiently computed through a fast Fourier transform. See [53] for more details.

II.4.2 Estimation of the approximation error

When the moving average process is localisable, or more precisely when the conditions of Proposition II.5 are satisfied, it is easy to assess the performances of the above synthesis method.

The following proposition gives a bound on the approximation error in the $\alpha-$norm. Recall that the $\alpha-$norm (defined in (I.3)) is just the scale factor of the random variable, and is thus independent of the integral representation that is used. In addition, it is, up to a constant depending on r and α, equals to the moments of order $0 < r < \alpha$.

Proposition II.12. Let Y be defined by (II.1), and let $Y_{\omega,\Omega}$ be its approximation defined in (II.20). Assume g satisfies the conditions of Proposition II.5. Then, for all $\omega, \Omega \in \mathbb{N}$ and $k \in \mathbb{Z}$ with $\omega > \frac{1}{\eta}$, one has

$$Err := \|Y(k) - Y_{\omega,\Omega}(k)\|_\alpha \leq A_{\omega,\Omega}^{1/\alpha}$$

where

$$A_{\omega,\Omega} = \frac{2c^\alpha}{(1+a\alpha)\omega^{1+\gamma\alpha}} \sum_{j=1}^{\omega\Omega} \frac{1}{j^{(a-\gamma)\alpha}} + \int_{-\infty}^{-\Omega} |g(s)|^\alpha ds + \int_{\Omega}^{+\infty} |g(s)|^\alpha ds$$

Proof

By stationarity and independence of the increments of Lévy motion, one gets :

$$Err = \sum_{j=-\omega\Omega+1}^{0} \int_{(j-1)/\omega}^{j/\omega} |g(\frac{j-1}{\omega}) - g(s)|^\alpha ds + \sum_{j=1}^{\omega\Omega} \int_{(j-1)/\omega}^{j/\omega} |g(\frac{j}{\omega}) - g(s)|^\alpha ds$$
$$+ \int_{-\infty}^{-\Omega} |g(s)|^\alpha ds + \int_{\Omega}^{+\infty} |g(s)|^\alpha ds. \tag{II.21}$$

By assumption, for almost all $s \in \mathbb{R}$, $|g(s+h) - g(s)| \leq c|h|^a|s|^{\gamma-a}$ when $0 < h < \eta$. Recall that $\omega > \frac{1}{\eta}$. A change of variables yields

$$\int_{(j-1)/\omega}^{j/\omega} |g(\frac{j-1}{\omega}) - g(s)|^\alpha ds \leq \int_0^{1/\omega} c^\alpha |s|^{a\alpha} |\frac{j-1}{\omega}|^{(\gamma-a)\alpha} ds$$

and thus

$$Err \leq \sum_{j=-\omega\Omega+1}^{0} \int_0^{1/\omega} c^\alpha |s|^{a\alpha} |\frac{j-1}{\omega}|^{(\gamma-a)\alpha} ds + \sum_{j=1}^{\omega\Omega} \int_0^{1/\omega} c^\alpha |s|^{a\alpha} |\frac{j}{\omega}|^{(\gamma-a)\alpha} ds$$
$$+ \int_{-\infty}^{-\Omega} |g(s)|^\alpha ds + \int_{\Omega}^{+\infty} |g(s)|^\alpha ds.$$

Rearranging terms :

$$\begin{aligned} Err &\leq \frac{c^\alpha}{(1+a\alpha)}\frac{1}{\omega^{1+\gamma\alpha}}\left(\sum_{j=-\omega\Omega+1}^{0}|j-1|^{(\gamma-a)\alpha}+\sum_{j=1}^{\omega\Omega}|j|^{(\gamma-a)\alpha}\right)\\ &+ \int_{-\infty}^{-\Omega}|g(s)|^\alpha ds+\int_{\Omega}^{+\infty}|g(s)|^\alpha ds\\ &= A_{\omega,\Omega}. \end{aligned}$$

which is the stated result ■

Corollary II.13. Under the conditions of Proposition II.12, $\|Y(k)-Y_{\omega,\Omega}(k)\|_\alpha \to 0$ when (ω,Ω) tends to infinity.

If in addition $g(x)\leq C|x|^{-\beta}$ when $|x|\to\infty$ for some $C>0$ and $\beta>\frac{1}{\alpha}$, then :

$$\|Y(k)-Y_{\omega,\Omega}(k)\|_\alpha^\alpha \leq K\left(\omega^{-1-\alpha\gamma}+\Omega^{1-\alpha\beta}\right) \tag{II.22}$$

where K is a constant independent of k,ω,Ω.

Proof

Since g satisfies the assumptions of Proposition II.5, $a>\gamma+\frac{1}{\alpha}$. As a consequence, the sum in the first term of $A_{\omega,\Omega}$ converges when (ω,Ω) tends to infinity. The first statement then follows from the facts that $\alpha\gamma+1>0$ and $g\in\mathcal{F}_\alpha$. The second part follows by making the obvious estimates ■

The significance of (II.22) is that it allows us to tune ω and Ω to obtain an optimal approximation, provided a bound on the decay of g at infinity is known : optimal pairs (ω,Ω) are those for which the two terms in (II.22) are of the same order of magnitude. More precisely, if the value of β is sharp, the order of decay of the error will be maximal when $\Omega=\omega^{\frac{-1-\alpha\gamma}{1-\alpha\beta}}$. Note that the exponent $\frac{-1-\alpha\gamma}{1-\alpha\beta}$ is always positive, as expected. Intuitively, ω is related to the regularity of g (irregular g requires larger ω), while Ω is linked with the rate of decay of g at infinity.

For concreteness, let us apply these results to some specific processes :

Example II.14. (reverse Ornstein-Uhlenbeck process) Let Y be the reverse Ornstein-Uhlenbeck process defined in Proposition II.2. When $\alpha>1$, we may apply Proposition II.12 with $g(x)=\exp(x)\mathbb{1}(x\leq 0)$, $\gamma=0$, $a=1$, $c=2$, $\eta=1$. One gets, for $\omega>1$, $\Omega>1$,

$$A_{\omega,\Omega}=\frac{2^{\alpha+1}}{1+\alpha}\left(\sum_{j=1}^{\omega\Omega}\frac{1}{j^\alpha}\right)\frac{1}{\omega}+\frac{e^{-\alpha\Omega}}{\alpha}.$$

However, we may obtain a more precise bound on the approximation error, valid for any $\alpha\in(0,2)$, by using (II.21) directly :

$$\|Y(k)-Y_{\omega,\Omega}(k)\|_\alpha^\alpha \leq \frac{2^\alpha}{1+\alpha}\left(\frac{1-e^{-\alpha\Omega}}{e^{\alpha/\omega}-1}\right)\frac{1}{\omega^{1+\alpha}}+\frac{e^{-\alpha\Omega}}{\alpha}.$$

When $(\omega,\Omega)\to+\infty$, $Err \leq \mathcal{O}(\frac{1}{\omega^\alpha})+\mathcal{O}(e^{-\alpha\Omega})$, which is better than $A_{\omega,\Omega}$ above when $\alpha>1$.

We note finally that the optimal choice for (ω,Ω) is here $\Omega=\ln(\omega)$, which is consistent with the fact that the β in Corollay II.13 may be chosen arbitrarily large.

Example II.15. (linear fractional stable noise) Let $0<\alpha\leq 2$ and let M be an α-stable symmetric random measure on $\mathbb{R}$ with control measure $\mathcal{L}$. Let :

$$g(x)=(x)_+^{H-1/\alpha}-(x-1)_+^{H-1/\alpha}$$

and

$$Y(t)=\int g(t-x)M(dx) \quad (t\in\mathbb{R})$$

Applying the analysis above with $\gamma=H-\frac{1}{\alpha}$, $a=1$,$c=2|\gamma|$,$\eta=1$ one gets, for $\omega>1$, $\Omega>1$,

$$A_{\omega,\Omega}=\frac{2^{\alpha+1}|H-\frac{1}{\alpha}|^\alpha}{1+\alpha}(\sum_{j=1}^{\omega\Omega}\frac{1}{j^{1+\alpha(1-H)}})\frac{1}{\omega^{\alpha H}}+\int_\Omega^{+\infty}|(x)^{H-1/\alpha}-(x-1)^{H-1/\alpha}|^\alpha dx$$

When $(\omega,\Omega)\to+\infty$,

$$\begin{aligned}A_{\omega,\Omega} &= \mathcal{O}(\frac{1}{\omega^{\alpha H}})+\int_\Omega^{+\infty}|(x)^{H-1/\alpha}-(x-1)^{H-1/\alpha}|^\alpha dx\\ &= \mathcal{O}(\frac{1}{\omega^{\alpha H}})+\mathcal{O}(\Omega^{1+\alpha(H-1/\alpha-1)})\\ &= \mathcal{O}(\frac{1}{\omega^{\alpha H}})+\mathcal{O}(\frac{1}{\Omega^{\alpha(1-H)}})\end{aligned}$$

This process is the one considered in [53]. Here we reach a conclusion similar to [53, Theorem 2.1], which yields the same order of magnitude for the error when $(\omega,\Omega)\to+\infty$. Extensive tests are conducted in [53] to choose the best values for (ω,Ω). The criterion for optimizing these parameters is to test how an estimation method for H performs on synthesized traces. Here we adopt a different approach based on Corollary II.13 : optimal pairs (ω,Ω) are those for which (II.22) is minimized. Since the value of $\beta=1-H+1/\alpha$ is sharp here, one gets $\Omega=\omega^{\frac{H}{1-H}}$. It is interesting to note that the exponent $H/(1-H)$ depends only on the scaling factor H and not on α, and that it may be larger or smaller than one depending on the value of H. We do not have an explanation for this fact nor for the reason why $H=1/2$ plays a special rôle.

Example II.16. As a final illustration, we consider the process of Example II.7. With $\gamma=\frac{1}{6}$, $a=1$,$c=1$,$\eta=1$, one gets, for $\omega>1$, $\Omega>1$,

$$A_{\omega,\Omega}=\frac{2}{1+\alpha}\left(\sum_{j=1}^{\omega\Omega}\frac{1}{j^{\frac{5\alpha}{6}}}\right)\frac{1}{\omega^{1+\frac{\alpha}{6}}}+\frac{1}{\frac{5\alpha}{6}-1}\frac{1}{\Omega^{\frac{5\alpha}{6}-1}}.$$

Again, the value of $\beta = 5/6$ is sharp, and the optimal choice is to set $\Omega = \omega^{\frac{\alpha+6}{5\alpha-6}}$. Since $(\alpha+6)/(5\alpha-6) \geq 1$, Ω is larger than ω in this case, in contrast to the reverse Ornstein-Uhlenbeck process : it is the decay at infinity of the kernel that dictates the parameters here, while it was the regularity that mattered in the case of the reverse Ornstein-Uhlenbeck process.

II.4.3 Numerical experiments

We display in Figure II.1 traces of :

- moving average stable processes : the reverse Ornstein-Uhlenbeck process (Figures 1(e),(f)), and the processes of Examples II.7 (figure 1(c)) and II.8 (figure 1(a)). In each case, $\alpha = 1.8$. Some of the relevant features of the processes of Examples II.7 and II.8 seem to appear more clearly when one integrates them. Thus integral versions are displayed in the right-hand part of the corresponding graphs, Figures 1(b),(d).
- a multistable version of the reverse Ornstein-Uhlenbeck process, using the theory developed in Section II.3 (Figures 1(g),(h)). Since these processes are localisable, one may obtain paths by computing first stable versions with all values assumed by α, and then "gluing" these tangent processes together as appropriate. More precisely, assume we want to obtain, at the discrete points $(t_1, \ldots, t_n)$, the values of a multistable process Y defined by a random field X. We first synthesize the n stable processes $X(., t_j)$ with the method described before, all are simulated from the same random seed because the three series $(\gamma_i)_i$, $(V_i)_i$ and $(\Gamma_i)_i$ are common for all the t_j. The multistable process Y is then obtained by setting $Y(t_i) = X(t_i, t_i), i = 1, \ldots, n$. Two graphs are displayed for the multistable process : in figure 1(g) the graphs are as explained above. In figure 1(h) each "line" $X(., t_i)$ of the random field (*i.e.* the process obtained for a fixed value of α) is renormalized so that it ranges between -1 and 1, prior to building the multistable process by gluing the paths as appropriate.

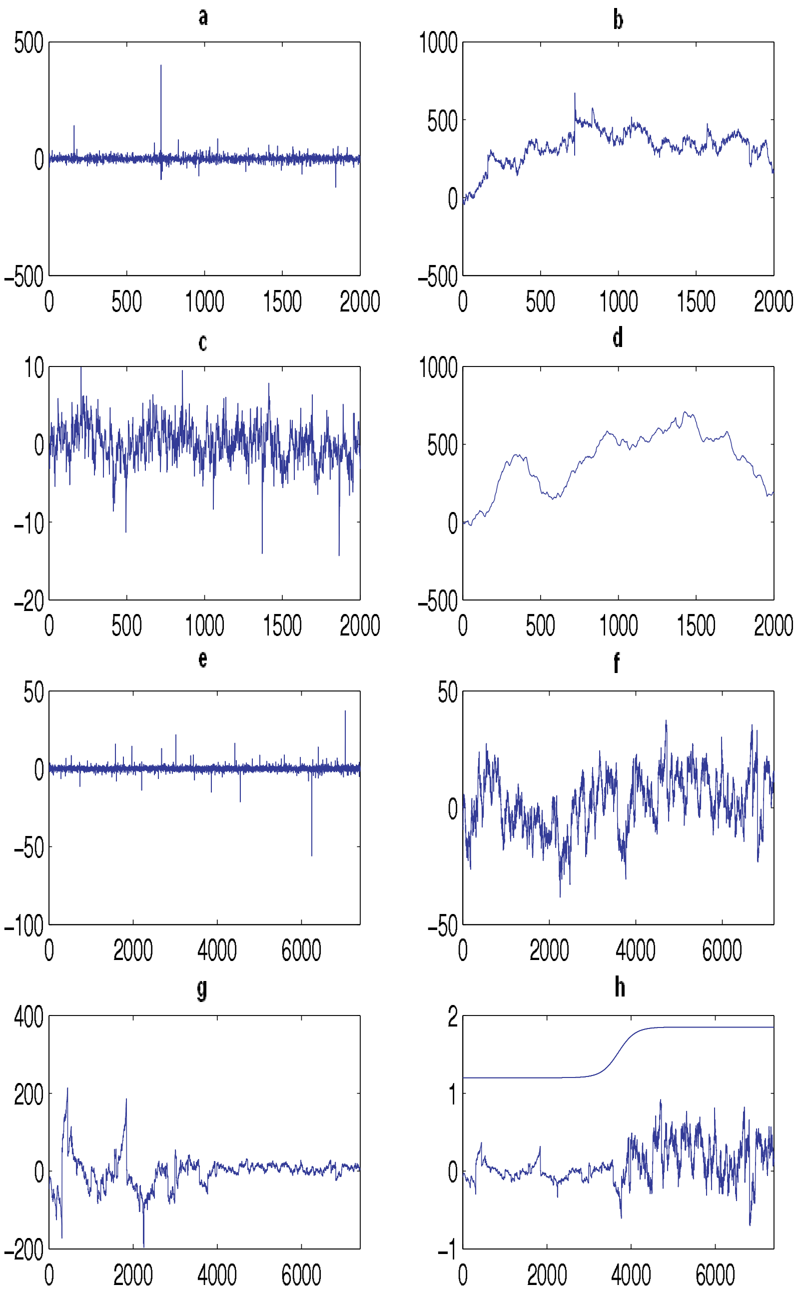


FIGURE II.1 – Paths of localisable processes. (a) The process in Example II.8 and (b) the integrated version. (c) The process in Example II.7 and (d) the integrated version. (e) Reverse Ornstein-Uhlenbeck processes with $\lambda = 1$, and (f) $\lambda = 0.01$. (g) A multistable reverse Ornstein-Uhlenbeck process with $\lambda = 0.01$ and (h) the renormalised version along with $\alpha(t)$.

The parameters are as follows :

- Process of Example II.8 : $\omega = 5000$, $\Omega = 877$, $N = 2000$. The approximation error Err is bounded by 2.172.
- Process of Example II.7 : $\omega = 104, \Omega = 175504$, $N = 2000$. The term $A_{\omega,\Omega}^{1/\alpha}$ is equal to 0.074.
- Reverse Ornstein-Uhlenbeck process with $\lambda = 1$: $\omega = 512, \Omega = 7, N = 7392$. The term $A_{\omega,\Omega}^{1/\alpha}$ is equal to 0.0018.
- Reverse Ornstein-Uhlenbeck process with $\lambda = 0.01$: $\omega = 256, \Omega = 800, N = 7392$. The term $A_{\omega,\Omega}^{1/\alpha}$ is equal to 0.0032.
- Multistable reverse Ornstein-Uhlenbeck process : $\lambda = 0.01, \omega = 256, \Omega = 800, N = 7392$. The α function is the logistic function starting from 1.2 and ending at 1.85. More precisely, we take : $\alpha(t) = 1.2 + \frac{0.65}{1+\exp(-\frac{5}{1000}(t-N/2))}$, where N is the number of points and t ranges from 1 to N (the graph of $\alpha(t)$ is plotted in figure 1(h)). Thus, one expects to see large jumps at the beginning of the paths and smaller ones at the end. Note that we do not have any results concerning the approximation error for these non-stationary processes.

The value of Ω in all cases is adjusted so that the pair (ω, Ω) is approximately "optimal" as described in the preceding subsection (optimality is not guaranteed for the multistable processes. Nevertheless, since the relation between ω and Ω does not depend on α for the reverse Ornstein-Uhlenbeck process, it holds in this case).

The function g of Example II.8 cannot be treated using Corollary II.13 nor Proposition II.12 since g does not satisfy the conditions of Proposition II.5. However, it is possible to estimate Err directly. Since $|g(x+h)-g(x)| \leq 2|h|(|x|^{-\frac{3}{2}}\mathbf{1}_{|x|<1}+|x|^{-1}\mathbf{1}_{|x|\geq 1})$ and $|g(x)| \leq \frac{4}{|x|}$, one gets :

$$Err^{\alpha} \leq \frac{2^{\alpha+2}}{1+\alpha}\sum_{j=1}^{\omega\Omega}\frac{1}{j^{\alpha}}\frac{1}{\omega^{1-\frac{\alpha}{2}}} + \frac{8}{\alpha-1}\frac{1}{\Omega^{\alpha-1}}$$

$$Err^{\alpha} \leq \frac{2^{\alpha+2}}{1+\alpha}\frac{\alpha}{\alpha-1}\frac{1}{\omega^{1-\frac{\alpha}{2}}} + \frac{8}{\alpha-1}\frac{1}{\Omega^{\alpha-1}}.$$

The asymptotic optimal relation between ω and Ω is thus $\Omega = \omega^{\frac{2-\alpha}{2(\alpha-1)}} = \omega^{0.125}$. The values in our simulation are slightly different since they are chosen to optimize the actual expression with a finite ω.

Finally, we stress that the same random seed (*i.e.* the same underlying stable $M(dx)$) has been used for all simulations, for easy comparison. Thus, for instance, the jumps appear at precisely the same locations in each graph. Notice in particular the ranges assumed by the different processes.

The differences between the graphs of the processes of Examples II.7, II.8 and the reverse Ornstein-Uhlenbeck process are easily interpreted by examining the three kernels : the kernel of the process of Example II.8 diverges at 0, thus putting more emphasis on strong jumps, as seen on the picture, with more jaggy curves and an "antipersistent" behaviour. The kernel of the process of Example II.7, in contrast, is smooth at the origin. In addition, it has a slow decay. These features result in an overall smoother appearance and allow "trends" to appear in the paths. Finally, the kernel of the reverse Ornstein-Uhlenbeck process has a decay controlled by λ. For "large" λ (here, $\lambda = 1$), little averaging is done, and the resulting path is very irregular. For "small" λ (here, $\lambda = 0.01$), the kernel decays slowly and the paths look smoother (recall that, in the Gaussian case, the Ornstein-Uhlenbeck tends in distribution to white noise when λ tends to infinity, and to Brownian motion when λ tends to 0).

Chapitre III

Représentation en séries de Ferguson-Klass-LePage des processus multifractionnaires multistables

Plan du chapitre

Abstract

The study of non-stationary processes whose local form has controlled properties is a fruitful and important area of research, both in theory and applications. In [16], a particular way of constructing such processes was investigated, leading in particular to *multifractional multistable processes*, which were built using sums over Poisson processes. We present here a different construction of these processes, based on the Ferguson - Klass - LePage series representation of stable processes. We consider various particular cases of interest, including multistable Lévy motion, multistable reverse Ornstein-Uhlenbeck process, log-fractional multistable motion and linear multistable multifractional motion. We also show that the processes defined here have the same finite dimensional distributions as the corresponding processes constructed in [16]. Finally, we display numerical experiments showing graphs of synthesized paths of such processes.

In the sequel, we shall consider specific classes of random fields and use Theorem I.14 to build localisable processes with interesting local properties. As a particular case, we will study multifractional multistable processes, where both the local Hölder regularity and intensity of jumps will evolve in a controlled manner.

The remaining of this article is organized as follows : we first build localisable processes using a series representation that yields the necessary flexibility required for our purpose. We need to distinguish between the situations where the underlying space is finite (Section III.1), or merely $\sigma-$ finite (Section III.2). In each case, we define a random field depending on a "kernel" f, and give conditions on f ensuring localisability of the diagonal process. We then consider in Section III.3 some examples : multistable Lévy motion, multistable reverse Ornstein-Uhlenbeck process, log-fractional multistable motion and linear multistable multifractional motion. Section III.4 is devoted to computing the finite dimensional distributions of our processes, and proving that they are the same as the ones of the corresponding processes constructed in [16]. Finally, Section III.5 displays graphs of certain localisable processes of interest, in particular multifractional multistable ones.

Before we proceed, we note that constructing localisable processes using a stochastic field composed of sssi processes is obviously not the only approach that one can think of. It is for instance possible to follow a rather different path and construct localisable processes from moving average ones by imposing conditions on the kernel defining the moving average. See [15] for details.

III.1 A Ferguson - Klass - LePage series representation of localisable processes in the finite measure space case

A well-known representation of stable random variables is the Ferguson - Klass - LePage series one [4, 18, 32, 33, 48]. This representation is particularly adapted for our purpose since, as we shall see, it allows for easy generalization to the case of varying α.

In this work, we will use the following version :

Theorem III.1. *([49, Theorem 3.10.1])*

Let $(E, \mathcal{E}, m)$ be a finite measure space where $m \not\equiv 0$, and M be a symmetric α-stable random measure with $\alpha \in (0,2)$ and finite control measure m. Let $(\Gamma_i)_{i\geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $\hat{m} = m/m(E)$ on E, and $(\gamma_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Assume finally that the three sequences $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, and $(\gamma_i)_{i\geq 1}$ are independent. Then, for any $f \in \mathcal{F}_\alpha(E, \mathcal{E}, m)$,

$$\int_E f(x) M(dx) \overset{d}{=} (C_\alpha m(E))^{1/\alpha} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha} f(V_i), \tag{III.1}$$

where $C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin(x) dx\right)^{-1}$ (Theorem 3.10.1 in [49] is more general, as it extends to non-symmetric stable processes, that are not considered here). As mentioned above, a relevant feature of this representation for us is that the distributions of all random variables appearing in the sum are independent of α. We will use (III.1) to construct processes with varying α as described in the following theorem.

Theorem III.2. *Let $(E, \mathcal{E}, m)$ be a finite measure space where $m \not\equiv 0$. Let α be a C^1 function defined on $\mathbb{R}$ and ranging in $(0, 2)$. Let b be a C^1 function defined on $\mathbb{R}$. Let $f(t, u, .)$ be a family of functions such that, for all $(t, u) \in \mathbb{R}^2$, $f(t, u, .) \in \mathcal{F}_{\alpha(u)}(E, \mathcal{E}, m)$. Let $(\Gamma_i)_{i\geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $\hat{m} = m/m(E)$ on E, and $(\gamma_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Assume finally that the three sequences $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, and $(\gamma_i)_{i\geq 1}$ are independent. Consider the following random field :*

$$X(t, u) = b(u)(m(E))^{1/\alpha(u)} C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} f(t, u, V_i), \tag{III.2}$$

where $C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin(x)dx\right)^{-1}$. Assume that $X(., u)$ is localisable at u with exponent $h \in (0, 1)$ and local form $X'_u(., u)$. Assume in addition that :

- *(C1) The family of functions $v \to f(t, v, x)$ is differentiable for all (v, t) in a neighbourhood of u and almost all x in E. The derivatives of f with respect to v are denoted by f'_v.*
- *(C3) There exists $\varepsilon > 0$ such that :*

$$\sup_{t\in B(u,\varepsilon)} \int_E \sup_{w\in B(u,\varepsilon)} (|f'_v(t, w, x)|^{\alpha(w)})\, \hat{m}(dx) < \infty. \tag{III.3}$$

- *(C4) There exists $\varepsilon > 0$ such that :*

$$\sup_{t\in B(u,\varepsilon)} \int_E \sup_{w\in B(u,\varepsilon)} \left[|f(t, w, x) \log |f(t, w, x)||^{\alpha(w)}\right]\, \hat{m}(dx) < \infty. \tag{III.4}$$

Then $Y(t) \equiv X(t, t)$ is localisable at u with exponent h and local form $Y'_u(t) = X'_u(t, u)$.

Proof

First, note that the condition (C4) implies the following condition :

- (C2) There exists $\varepsilon > 0$ such that :

$$\sup_{t\in B(u,\varepsilon)} \int_E \sup_{w\in B(u,\varepsilon)} (|f(t, w, x)|^{\alpha(w)})\, \hat{m}(dx) < \infty. \tag{III.5}$$

Indeed, for all $t \in B(u, \varepsilon)$, $w \in B(u, \varepsilon)$ and $x \in E$,

$$\begin{aligned} |f(t, w, x)| &= |f(t, w, x)|\mathbf{1}_{|f(t,w,x)|<\frac{1}{e}} + |f(t, w, x)|\mathbf{1}_{|f(t,w,x)|>e} + |f(t, w, x)|\mathbf{1}_{|f(t,w,x)|\in[\frac{1}{e},e]} \\ &\leq 2|f(t, w, x)|| \log |f(t, w, x)|| + e. \end{aligned}$$

The function $u \mapsto C_{\alpha(u)}^{1/\alpha(u)}$ is C^1 since $\alpha(u)$ ranges in $(0, 2)$. We shall denote $a(u) = b(u)(m(E))^{1/\alpha(u)} C_{\alpha(u)}^{1/\alpha(u)}$. The function a is thus also C^1. We want to apply Theorem I.14. With that in view, we estimate, for $v \in B(u, \varepsilon)$ (the ball centered at u with radius ε),

$$X(v, v) - X(v, u) =: \sum_{i=1}^{\infty} \gamma_i(\Phi_i(v) - \Phi_i(u)) + \sum_{i=1}^{\infty} \gamma_i(\Psi_i(v) - \Psi_i(u)),$$

where

$$\Phi_i(w) = a(w) i^{-1/\alpha(w)} f(v, w, V_i)$$

and

$$\Psi_i(w) = a(w) \left(\Gamma_i^{-1/\alpha(w)} - i^{-1/\alpha(w)}\right) f(v, w, V_i).$$

The reason for introducing the Φ_i and the Ψ_i is that the random variables Γ_i are not independent, which complicates their study. We shall decompose the sum involving the Φ_i into series of independent random variables which will be dealt with using the three series theorem. The sum involving the Ψ_i will be studied by taking advantage of the fact that, for large enough i, each Γ_i is "close" to i in some sense.

Let $c = \inf_{v \in B(u,\varepsilon)} \alpha(v)$, $d = \sup_{v \in B(u,\varepsilon)} \alpha(v)$. If $\inf_{v \in B(u,\varepsilon)} \alpha(v) = \sup_{v \in B(u,\varepsilon)} \alpha(v)$, we let instead $c = \inf_{v \in B(u,\varepsilon)} \alpha(v)$, $d = c + \varepsilon$, for some $\varepsilon > 0$. Note that, in both cases, by decreasing ε, $d - c$ may be made arbitrarily small.

Thanks to the assumptions on a and f, Φ_i and Ψ_i are differentiable almost surely and one computes :

$$\Phi_i'(w) = a'(w) i^{-1/\alpha(w)} f(v, w, V_i) + a(w) i^{-1/\alpha(w)} f_w'(v, w, V_i) + a(w) \frac{\alpha'(w)}{\alpha(w)^2} \log(i) i^{-1/\alpha(w)} f(v, w, V_i),$$

and

$$\Psi_i'(w) = a'(w) \left(\Gamma_i^{-1/\alpha(w)} - i^{-1/\alpha(w)}\right) f(v, w, V_i) + a(w) \left(\Gamma_i^{-1/\alpha(w)} - i^{-1/\alpha(w)}\right) f_w'(v, w, V_i)$$
$$+ a(w) \frac{\alpha'(w)}{\alpha(w)^2} \left(\log(\Gamma_i) \Gamma_i^{-1/\alpha(w)} - \log(i) i^{-1/\alpha(w)}\right) f(v, w, V_i).$$

Notice that the functions Φ_i' and Ψ_i' depend on v.

Consider now the function $h_i : x \to \Phi_i(x) - \Phi_i(u) - \dfrac{\Phi_i(v) - \Phi_i(u)}{v - u}(x - u)$, and the set $K_i = \{x \in [u, v] : h_i'(x) = 0\}$.

The mean value theorem yields that K_i is a non-empty closed set of $\mathbb{R}$. We define then

$$w_i = \min K_i.$$

Considering the function $k_i : x \to \Psi_i(x) - \Psi_i(u) - \dfrac{\Psi_i(v) - \Psi_i(u)}{v - u}(x - u)$, and the set $F_i = \{x \in [u, v] : k_i'(x) = 0\}$, we define also

$$x_i = \min F_i.$$

Then there exists a sequence of independent measurable random numbers $w_i \in [u,v]$ (or $[v,u]$) and a sequence of measurable random numbers $x_i \in [u,v]$ (or $[v,u]$) such that :

$$X(v,u) - X(v,v) = (u-v)\sum_{i=1}^{\infty}(Z_i^1 + Z_i^2 + Z_i^3) + (u-v)\sum_{i=1}^{\infty}(Y_i^1 + Y_i^2 + Y_i^3),$$

where

$$\begin{aligned}
Z_i^1 &= \gamma_i a'(w_i) i^{-1/\alpha(w_i)} f(v,w_i,V_i),\\
Z_i^2 &= \gamma_i a(w_i) i^{-1/\alpha(w_i)} f'_u(v,w_i,V_i),\\
Z_i^3 &= \gamma_i a(w_i)\frac{\alpha'(w_i)}{\alpha(w_i)^2}\log(i) i^{-1/\alpha(w_i)} f(v,w_i,V_i),\\
Y_i^1 &= \gamma_i a'(x_i)\left(\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}\right) f(v,x_i,V_i),\\
Y_i^2 &= \gamma_i a(x_i)\left(\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}\right) f'_u(v,x_i,V_i),\\
Y_i^3 &= \gamma_i a(x_i)\frac{\alpha'(x_i)}{\alpha(x_i)^2}\left(\log(\Gamma_i)\Gamma_i^{-1/\alpha(x_i)} - \log(i) i^{-1/\alpha(x_i)}\right) f(v,x_i,V_i).
\end{aligned}$$

Note that each w_i depends on a, f, α, u, v, V_i, but not on γ_i. This remark will be useful in the sequel.

We establish now a lemma in order to control the series $\sum_{i=1}^{\infty} \mathsf{P}(|Z_i^1| > \lambda)$.

Lemma III.3. There exists a positive constant K such that for all $\lambda > 0$,

$$\sum_{i=1}^{\infty}\mathsf{P}(|Z_i^1| > \lambda) \leq \frac{K\mathsf{E}\left[\sup_{w\in B(u,\varepsilon)}|f(v,w,V_1)|^{\alpha(w)}\right]}{\min(\lambda^c,\lambda^d)}.$$

Proof of Lemma III.3
Fix $\lambda > 0$.

$$\begin{aligned}
\mathsf{P}(|Z_i^1| > \lambda) &= \mathsf{P}\left(|f(v,w_i,V_i)| > \frac{\lambda i^{1/\alpha(w_i)}}{|a'(w_i)|}\right)\\
&\leq \mathsf{P}\left(|f(v,w_i,V_i)|^{\alpha(w_i)} > i\min(\lambda^c,\lambda^d)\inf_{w\in B(u,\varepsilon)}\left[\frac{1}{|a'(w)|^{\alpha(w)}}\right]\right).
\end{aligned}$$

Note that, since a' is bounded on the compact interval $[u,v]$, $K := \inf_{w\in B(u,\varepsilon)}\left[\frac{1}{|a'(w)|^{\alpha(w)}}\right]$ is strictly positive.

$$\begin{aligned}
\mathsf{P}(|Z_i^1| > \lambda) &\leq \mathsf{P}\left(\sup_{w\in B(u,\varepsilon)}|f(v,w,V_i)|^{\alpha(w)} > Ki\min(\lambda^c,\lambda^d)\right)\\
&= \mathsf{P}\left(\sup_{w\in B(u,\varepsilon)}|f(v,w,V_1)|^{\alpha(w)} > Ki\min(\lambda^c,\lambda^d)\right).
\end{aligned}$$

Thus

$$\begin{aligned}\sum_{i=1}^{+\infty} \mathsf{P}(|Z_i^1| > \lambda) &\leq \sum_{i=1}^{+\infty} \mathsf{P}\left(\sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)} > Ki\min(\lambda^c,\lambda^d)\right) \\ &\leq \frac{1}{K\min(\lambda^c,\lambda^d)}\mathsf{E}\left[\sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)}\right] \quad \blacksquare\end{aligned}$$

Now we come back to the proof of Theorem III.2. The remainder of the proof is divided into four steps. The first step will apply the three-series theorem to show that each series $\sum_{i=1}^{\infty} Z_i^j, j = 1,2,3$, converges almost surely. In the second step, we will prove that $\sum_{i=1}^{\infty} Y_i^j$ also converges almost surely for $j = 1,2,3$. In the third step we will prove that condition (I.12) is verified by $\sum_{i=1}^{\infty} Z_i^j, j = 1,2,3$. Finally, step four will prove the same thing for $\sum_{i=1}^{\infty} Y_i^j, j = 1,2,3$.

First step : almost sure convergence of $\sum_{i=1}^{\infty} Z_i^j, j = 1,2,3$.

Consider $Z^1 = \sum_{i=1}^{\infty} Z_i^1$. Fix $\lambda > 0$. We shall deal successively with the three series involved the three-series theorem.

First series : $S_1 = \sum_{i=1}^{\infty} \mathsf{P}(|Z_i^1| > \lambda)$.

One has $S_1 < +\infty$ from Lemma III.3 and the condition (C2).

Second series : $S_2^n = \sum_{i=1}^{n} \mathsf{E}(Z_i^1 \mathbb{1}\{|Z_i^1| \leq \lambda\})$.

$$\begin{aligned}\mathsf{E}(Z_i^1 \mathbb{1}\{|Z_i^1| \leq \lambda\}) &= \mathsf{E}(\gamma_i a'(w_i) i^{-1/\alpha(w_i)} f(v,w_i,V_i)\mathbb{1}\{|a'(w_i)i^{-1/\alpha(w_i)} f(v,w_i,V_i)| \leq \lambda\}) \\ &= \mathsf{E}(\gamma_i)\mathsf{E}(a'(w_i)i^{-1/\alpha(w_i)} f(v,w_i,V_i)\mathbb{1}\{|a'(w_i)i^{-1/\alpha(w_i)} f(v,w_i,V_i)| \leq \lambda\}) \\ &= 0,\end{aligned}$$

where we have used the facts that γ_i is independent of (w_i, V_i) and $\mathsf{E}(\gamma_i) = 0$. As a consequence, $\lim_{n\to+\infty} S_2^n = 0$.

Third series : The final series we need to consider is $S_3 = \sum_{i=1}^{\infty} \mathsf{E}\left[(Z_i^1 \mathbb{1}\{|Z_i^1| \leq 1\})^2\right]$. Chose $\lambda = 1$ [1].

Let η be such that $d < \eta < 2$.

1. Recall that, in the three series theorem, for the series $\sum_{i=1}^{\infty} X_i$ to converge almost surely, it is necessary that, for all $\lambda > 0$, the three series $\sum_{i=1}^{\infty} \mathsf{P}(|X_i| > \lambda)$, $\sum_{i=1}^{\infty} \mathsf{E}(X_i\mathbb{1}(|X_i| \leq \lambda))$, and $\sum_{i=1}^{\infty} \mathrm{Var}(X_i\mathbb{1}(|X_i| \leq \lambda))$ converge, and it is sufficient that they converge for *one* $\lambda > 0$, see, e.g. [42], Theorem 6.1.

$$\begin{aligned}
(Z_i^1 \mathbf{1}\{|Z_i^1| \leq 1\})^2 &\leq |Z_i^1|^\eta \mathbf{1}\{|Z_i^1| \leq 1\} \\
\mathsf{E}\left[(Z_i^1 \mathbf{1}\{|Z_i^1| \leq 1\})^2\right] &\leq \mathsf{E}\left[|Z_i^1|^\eta \mathbf{1}\{|Z_i^1| \leq 1\}\right] \\
&= \int_0^{+\infty} \mathsf{P}(|Z_i^1|^\eta \mathbf{1}\{|Z_i^1| \leq 1\} > x)dx \\
&= \int_0^1 \mathsf{P}(|Z_i^1|^\eta \mathbf{1}\{|Z_i^1| \leq 1\} > x)dx \\
&\leq \int_0^1 \mathsf{P}(|Z_i^1|^\eta > x)dx.
\end{aligned}$$

Now from Lemma III.3 and (C2), there exists a positive constant K such that

$$\begin{aligned}
S_3 &\leq K \int_0^1 \frac{dx}{\min(x^{\frac{c}{\eta}}, x^{\frac{d}{\eta}})} \\
&= K \int_0^1 \frac{dx}{x^{\frac{d}{\eta}}} \\
&< +\infty.
\end{aligned}$$

The case of the $Z^2 = \sum_{i=1}^{\infty} Z_i^2$ is treated similarly, since the conditions required on (a', f) in the proof above are also satisfied by (a, f'_u).

Consider finally $Z^3 = \sum_{i=1}^{\infty} Z_i^3$.

Since the series $\sum_{i=1}^{\infty} \gamma_i(\Phi_i(v) - \Phi_i(u))$ converges almost surely (see for instance [34, page 132]), the convergence of Z^1 and Z^2 imply the convergence of Z^3.
We have thus shown that the series Z^1, Z^2 and Z^3 are almost surely convergent.

Second step : almost sure convergence of $\sum_{i=1}^{\infty} Y_i^j, j = 1, 2, 3$.

To prove that the series $\sum_{i=1}^{\infty} Y_i^j, j = 1, 2$ converge almost surely, we will first show that it is enough to prove that $\sum_{i=1}^{\infty} Y_i^j \mathbf{1}_{\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\} \cap \{|Y_i^j| \leq 1\}}$ converges almost surely for $j = 1, 2$. Indeed, we prove now that $\sum_{i=1}^{\infty} \mathsf{P}\left(\overline{\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}} \cup \{|Y_i^j| > 1\}\right) < \infty$ for $j = 1, 2$, where $\overline{T}$ denotes the complementary set of the set T, and conclude with the Borel Cantelli lemma. The case of $\sum_{i=1}^{+\infty} Y_i^3$ is then treated as $\sum_{i=1}^{+\infty} Z_i^3$. We know that the series $\sum_{i=1}^{\infty} \gamma_i(\Psi_i(v) - \Psi_i(u))$ converges almost surely (see again [34, page 132]), the convergence of $\sum_{i=1}^{+\infty} Y_i^1$ and $\sum_{i=1}^{+\infty} Y_i^2$ will imply the convergence of $\sum_{i=1}^{+\infty} Y_i^3$.

We have

$$\begin{aligned}
\mathsf{P}\left(\overline{\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}} \cup \{|Y_i^j| > 1\}\right) &= \mathsf{P}\left(\overline{\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}} \cup \left[\{|Y_i^j| > 1\} \cap \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}\right]\right) \\
&\leq \mathsf{P}(\Gamma_i < \frac{i}{2}) + \mathsf{P}(\Gamma_i > 2i) + \mathsf{P}\left(\{|Y_i^j| > 1\} \cap \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}\right).
\end{aligned}$$

Γ_i, as a sum of independent and identically distributed exponential random variables with mean 1, satisfy a Large Deviation Principle with rate function $\Lambda^*(x) = x - 1 - \log(x)$ for $x > 0$ and infinity for $x \leq 0$ (see for instance [11] p.35), thus $\sum_{i\geq 1} \mathsf{P}(\Gamma_i < \frac{i}{2}) < +\infty$ and $\sum_{i\geq 1} \mathsf{P}(\Gamma_i > 2i) < +\infty$.

Consider now $\sum_{i\geq 1} \mathsf{P}\left(\{|Y_i^j| > 1\} \cap \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}\right)$, for $j = 1, 2$.

<u>Case $j = 1$</u> :

Let $B_i = \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}$.

$$\begin{aligned}
\mathsf{P}\left(\{|Y_i^1| > 1\} \cap \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}\right) &= \mathsf{P}\left(\{|a'(x_i) i^{-1/\alpha(x_i)} f(v, x_i, V_i)||(\frac{\Gamma_i}{i})^{-1/\alpha(x_i)} - 1| > 1\} \cap B_i\right) \\
&\leq \mathsf{P}\left(\{(2^{1/\alpha(x_i)} - 1)|a'(x_i) i^{-1/\alpha(x_i)} f(v, x_i, V_i)| > 1\} \cap B_i\right) \\
&\leq \mathsf{P}\left(\sup_{w\in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)} > Ki\right)
\end{aligned}$$

where K is a positive constant. Thus $\sum_{i\geq 1} \mathsf{P}\left(\{|Y_i^1| > 1\} \cap \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}\right) < +\infty$.

<u>Case $j = 2$</u> : Since the conditions required on (a', f) in the proof above are also satisfied by (a, f'_u), $\sum_{i\geq 1} \mathsf{P}\left(\{|Y_i^2| > 1\} \cap \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}\right) < +\infty$.

We are thus left with proving that $\sum_{i=1}^{\infty} Y_i^j \mathbf{1}_{\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\} \cap \{|Y_i^j| \leq 1\}}$ converges almost surely for $j = 1, 2$.

In that view, we shall apply the following well-known lemma :

Lemma III.4. Let $\{X_k, k \geq 1\}$ be a sequence of random variables such that $\sum_{n=1}^{+\infty} \mathsf{E}|X_n| < +\infty$, then $\sum_{n=1}^{+\infty} X_n$ converges almost surely.

Let us show that $\sum_{i=1}^{\infty} \mathsf{E}\left[|Y_i^j|\mathbf{1}_{\{\frac{1}{2}\leq\frac{\Gamma_i}{i}\leq 2\}\cap\{|Y_i^j|\leq 1\}}\right] < +\infty$.

$$\begin{aligned}
\mathsf{E}\left[|Y_i^j|\mathbf{1}_{\{\frac{1}{2}\leq\frac{\Gamma_i}{i}\leq 2\}\cap\{|Y_i^j|\leq 1\}}\right] &= \int_0^{\infty} \mathsf{P}\left(\{1 \geq |Y_i^j| > x\} \cap \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}\right) dx \\
&\leq \int_0^{1} \mathsf{P}\left(\{|Y_i^j| > x\} \cap \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}\right) dx.
\end{aligned}$$

Let $B_i = \{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}$.

Case $j = 1$:

Using the finite-increments formula applied to the function $y \mapsto y^{-\frac{1}{\alpha(x_i)}}$ on $[\frac{1}{2}, 2]$, one easily shows that

$$\begin{aligned}
\mathsf{P}\left(\{|Y_i^1| > x\} \cap B_i\right) &\leq \mathsf{P}\left(|a'(x_i)i^{-1/\alpha(x_i)} f(v, x_i, V_i)||\frac{\Gamma_i}{i} - 1| > x\frac{c}{2^{1+1/c}}\} \cap B_i\right) \\
&\leq \mathsf{P}\left(\{|f(v, x_i, V_i)|^{\alpha(x_i)}|\frac{\Gamma_i}{i} - 1|^{\alpha(x_i)} > K_c i x^{\alpha(x_i)}\} \cap B_i\right)
\end{aligned}$$

where $K_c := \inf_{w\in B(u,\varepsilon)} \left[\left(\frac{c}{2^{1+1/c}|a'(w)|}\right)^{\alpha(w)}\right]$ is strictly positive by the assumptions on a' and α. Thus, for $x \in (0, 1)$,

$$\mathsf{P}\left(\{|Y_i^1| > x\} \cap B_i\right) \leq \mathsf{P}\left(\sup_{w\in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)}|\frac{\Gamma_i}{i} - 1|^c > K_c i x^d\right).$$

Case $d \geq 1$:

Fix $\eta \in (d, 1 + \frac{c}{2})$ (since α is continuous and $d < 2$, by decreasing if necessary ε, one may ensure that $d < 1 + c/2$). By Markov and Hölder inequalities, and the independence of V_1 and Γ_i,

$$\begin{aligned}
\mathsf{P}\left(\{|Y_i^1| > x\} \cap B_i\right) &\leq \mathsf{P}\left(\left[\sup_{w\in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)}\right]^{1/\eta} |\frac{\Gamma_i}{i} - 1|^{c/\eta} > K_c^{1/\eta} i^{1/\eta} x^{d/\eta}\right) \\
&\leq \frac{1}{(K_c i x^d)^{1/\eta}} \left[\mathsf{E}|\frac{\Gamma_i}{i} - 1|^2\right]^{c/2\eta} \left(\sup_{v\in B(u,\varepsilon)} \mathsf{E}(\sup_{w\in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)})\right)^{1/\eta} \\
&\leq \frac{K}{x^{d/\eta}} \frac{1}{i^{1/\eta+c/2\eta}}
\end{aligned}$$

where we have used that the variance of Γ_i is equal to i, and K does not depend on v thanks to condition (C2). Thus $\mathsf{E}\left[|Y_i^1|\mathbf{1}_{\{\frac{1}{2}\leq\frac{\Gamma_i}{i}\leq 2\}\cap\{|Y_i^1|\leq 1\}}\right] \leq \frac{K}{i^{1/\eta+c/2\eta}}$ where $\frac{1}{\eta} + \frac{c}{2\eta} > 1$.

Case $d < 1$:

$$\begin{aligned} \mathsf{P}\left(\{|Y_i^1| > x\} \cap B_i\right) &\leq \frac{1}{x^d K_c i} \mathsf{E}(\sup_{w \in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)}) \mathsf{E}|\frac{\Gamma_i}{i} - 1|^c \\ &\leq K \frac{1}{x^d} \frac{1}{i} (\mathsf{E}|\frac{\Gamma_i}{i} - 1|^2)^{c/2} \\ &\leq K \frac{1}{i^{1+c/2}} \frac{1}{x^d}, \end{aligned}$$

thus $\mathsf{E}\left[|Y_i^1| \mathbf{1}_{\{\frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\} \cap \{|Y_i^1| \leq 1\}}\right] \leq \frac{K}{i^{1+c/2}}$ with $1 + \frac{c}{2} > 1$.

The case of $\sum\limits_{i \geq 1} \mathsf{E}\left[|Y_i^2| \mathbf{1}_{\{B_i \cap \{|Y_i^2| \leq 1\}}\right]$ is treated similarly, since the conditions required on (a', f) in the proof above are also satisfied by (a, f'_u).

As a conclusion, for $j = 1, 2, 3$, $\sum\limits_{i=1}^{+\infty} Y_i^j$ converges almost surely.

We now move to the last two steps of the proof : to verify h-localisability, we need to check that for some η such that $h < \eta < 1$, $\mathsf{P}(|\sum\limits_{i=1}^{\infty} Z_i^j| \geq |v-u|^{\eta-1})$ and $\mathsf{P}(|\sum\limits_{i=1}^{\infty} Y_i^j| \geq |v-u|^{\eta-1})$ tend to 0 when v tends to u, for $j = 1, 2, 3$.

Third step : verification of (I.12) for $\sum\limits_{i=1}^{\infty} Z_i^j, j = 1, 2, 3$.

We need to estimate $\mathsf{P}(|\sum\limits_{i=1}^{\infty} Z_i^j| \geq |v-u|^{\eta-1})$.

Let $a \in (0, 1-\eta)$.

$$\begin{aligned} \mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j| > |v-u|^{\eta-1}\right) &\leq \mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j \mathbf{1}_{|Z_i^j| \leq |v-u|^{-a}}| > \frac{|v-u|^{\eta-1}}{2}\right) \\ &+ \mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j \mathbf{1}_{|Z_i^j| > |v-u|^{-a}}| > \frac{|v-u|^{\eta-1}}{2}\right). \end{aligned}$$

Since γ_i is independent from γ_k for $i \neq k$ and $|Z_i^j|$ is independent of γ_i, Markov inequality yields

$$\mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j \mathbf{1}_{|Z_i^j| \leq |v-u|^{-a}}| > \frac{|v-u|^{\eta-1}}{2}\right) \leq \frac{4}{|v-u|^{2(\eta-1)}} \sum_{i=1}^{\infty} \mathsf{E}\left[|Z_i^j|^2 \mathbf{1}_{|Z_i^j| \leq |v-u|^{-a}}\right].$$

Let $\gamma \in (d, 2)$. For any $M > 1$, we get :

$$\mathsf{E}\left[|Z_i^j|^2 \mathbf{1}_{|Z_i^j|\leq M}\right] = M^2 \mathsf{E}\left[\frac{|Z_i^j|^2}{M^2} \mathbf{1}_{|Z_i^j|\leq M}\right] \leq M^2 \int_0^1 \mathsf{P}(|Z_i^j| > M x^{1/\gamma}) dx.$$

For $j=1$, we use again Lemma III.3 : there exists a positive constant K such that

$$\begin{aligned}\sum_{i=1}^{\infty} \mathsf{E}\left[|Z_i^1|^2 \mathbf{1}_{|Z_i^j|\leq M}\right] &\leq M^2 K \int_0^1 \frac{dx}{\min(M^c x^{c/\gamma}, M^d x^{d/\gamma})} \\ &\leq M^{2-c} \int_0^1 \frac{dx}{\min(x^{c/\gamma}, x^{d/\gamma})}.\end{aligned}$$

Thus, there exists a positive constant K such that

$$\sum_{i=1}^{\infty} \mathsf{E}\left[|Z_i^1|^2 \mathbf{1}_{|Z_i^1|\leq M}\right] \leq K M^{2-c}.$$

The same conclusion holds for $j=2$:

$$\sum_{i=1}^{\infty} \mathsf{E}\left[|Z_i^2|^2 \mathbf{1}_{|Z_i^2|\leq M}\right] \leq K M^{2-c}.$$

For $j=3$, choose $\mu \in (0, 1-\frac{d}{\gamma})$.
Fix $\lambda > 0$.

$$\begin{aligned}\mathsf{P}(|Z_i^3| > \lambda) &= \mathsf{P}\left(|f(v,w_i,V_i)| > \frac{\lambda \alpha(w_i)^2 i^{1/\alpha(w_i)}}{|a(w_i)\alpha'(w_i)| \log i}\right) \\ &\leq \mathsf{P}\left(|f(v,w_i,V_i)|^{\alpha(w_i)} > K'' \lambda^{\alpha(w_i)} \frac{i}{(\log i)^{\alpha(w_i)}}\right),\end{aligned}$$

where $K'' := \inf_{w\in B(u,\varepsilon)}\left[\left(\frac{\alpha(w)^2}{|a(w)\alpha'(w)|}\right)^{\alpha(w)}\right]$ is strictly positive by the assumptions on a, α and α'. In the sequel, K will always denote a finite positive constant, that may however change from line to line.

Let $g_i(x) = \frac{x}{(\log x)^{\alpha(w_i)}}$ for $x \geq 1$ and $i \in \mathbb{N}^*$. For x large enough and for all i, g_i is strictly increasing and $\lim_{x\to+\infty} g_i(x) = +\infty$. For z large enough (independently of i),

$$\begin{aligned}g_i(z(\log z)^{\alpha(w_i)}) &= \frac{z(\log z)^{\alpha(w_i)}}{(\log z + \alpha(w_i)\log\log z)^{\alpha(w_i)}} \\ &\geq \frac{z}{2}.\end{aligned}$$

Let $A > e$ be such that : $\forall z \geq A, \forall i \in \mathbb{N}^*, g_i^{-1}(z) \leq K z(\log z)^{\alpha(w_i)}$. The constant A depends only on α.

Let $U_i = |f(v,w_i,V_i)|^{\alpha(w_i)}$, and i^* depending only on α such that for all $i \geq i^*$, $\frac{i}{(\log i)^{\alpha(w_i)}} \geq A$. We have, for $i \geq i^*$,

$$\begin{aligned}
\mathsf{P}(|Z_i^3| > \lambda) &\leq \mathsf{P}\left(\frac{U_i}{K''\lambda^{\alpha(w_i)}} > \frac{i}{(\log i)^{\alpha(w_i)}}\right) \\
&\leq \mathsf{P}\left(i \leq \frac{KU_i}{K''\lambda^{\alpha(w_i)}}\left(\log(\frac{U_i}{K''\lambda^{\alpha(w_i)}})\right)^{\alpha(w_i)}\right) \\
&\leq \mathsf{P}\left(i \leq K\frac{U_i\log U_i}{\lambda^{\alpha(w_i)}} + K\frac{U_i}{\lambda^{\alpha(w_i)}} + K\frac{|\log\lambda|}{\lambda^{\alpha(w_i)}}U_i\right) \\
&\leq \mathsf{P}\left(\sup_{w\in B(u,\varepsilon)}[|f(v,w,V_1)\log|f(v,w,V_1)||^{\alpha(w)}] > Ki\min(\lambda^c,\lambda^d)\right) \\
&\quad +\mathsf{P}\left(\sup_{w\in B(u,\varepsilon)}|f(v,w,V_1)|^{\alpha(w)} > Ki\frac{\min(\lambda^c,\lambda^d)}{|\log\lambda|}\right).
\end{aligned}$$

Finally, with (C4), for $M > e$,

$$\begin{aligned}
\sum_{i=i^*}^{\infty}\int_0^1 \mathsf{P}(|Z_i^3| > Mx^{1/\gamma})dx &\leq KM^{-c} + K\int_0^1 \frac{|\log(Mx^{1/\gamma})|}{\min(M^c x^{c/\gamma}, M^d x^{d/\gamma})}dx \\
&\leq K\log(M)M^{-c}.
\end{aligned}$$

We get then

$$\sum_{i=i^*}^{\infty}\mathsf{E}\left[|Z_i^3|^2\mathbf{1}_{|Z_i^3|\leq M}\right] \leq K\log(M)M^{-c}.$$

Let $M = |v-u|^{-a}$. Using previously obtained inequalities, we get, for $j = 1,2,3$:

$$\mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j\mathbf{1}_{|Z_i^j|\leq|v-u|^{-a}}| > \frac{|v-u|^{\eta-1}}{2}\right) \leq K|v-u|^{2(1-\eta)-a(2-c)}\log|v-u| + K|v-u|^{2(1-\eta)-2a}$$

and

$$\lim_{v\to u}\mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j\mathbf{1}_{|Z_i^j|\leq|v-u|^{-a}}| > \frac{|v-u|^{\eta-1}}{2}\right) = 0.$$

We consider now the second term $\mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j\mathbf{1}_{|Z_i^j|>|v-u|^{-a}}| > \frac{|v-u|^{\eta-1}}{2}\right)$.

Let $i^* = \inf\{n \geq 1 : i \geq n, |Z_i^j| \leq |v-u|^{-a}\}$. Since $\sum_{i\geq 1}\mathsf{P}(|Z_i^j| > |v-u|^{-a}) < +\infty$, the Borel-Cantelli lemma yields $\mathsf{P}(i^* = +\infty) = 0$. As a consequence,

$$\begin{aligned}
\mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j \mathbf{1}_{|Z_i^j|>|v-u|^{-a}}| > \frac{|v-u|^{\eta-1}}{2}\right) &= \sum_{n=1}^{\infty} \mathsf{P}\left(\{|\sum_{i=1}^{\infty} Z_i^j \mathbf{1}_{|Z_i^j|>|v-u|^{-a}}| > \frac{|v-u|^{\eta-1}}{2}\} \cap \{i^*=n\}\right) \\
&= \sum_{n=2}^{\infty} \mathsf{P}\left(\{|\sum_{i=1}^{\infty} Z_i^j \mathbf{1}_{|Z_i^j|>|v-u|^{-a}}| > \frac{|v-u|^{\eta-1}}{2}\} \cap \{i^*=n\}\right) \\
&\leq \sum_{n=2}^{\infty} \mathsf{P}(i^*=n).
\end{aligned}$$

For $n \geq 2$, $\mathsf{P}(i^*=n) \leq \mathsf{P}(|Z_{n-1}^j| > |v-u|^{-a})$.
For $j=1$, $\mathsf{P}(i^*=n) \leq \mathsf{P}(\sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)} > |v-u|^{-ac} K(n-1))$, and thus

$$\begin{aligned}
\sum_{n=2}^{\infty} \mathsf{P}(i^*=n) &\leq K|v-u|^{ac} \mathsf{E}(\sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)}) \\
&\leq K|v-u|^{ac} \sup_{t\in B(u,\varepsilon)} \mathsf{E}(\sup_{w\in B(u,\varepsilon)} |f(t,w,V_1)|^{\alpha(w)}).
\end{aligned}$$

For $j=2$,

$$\sum_{n=2}^{\infty} \mathsf{P}(i^*=n) \leq K|v-u|^{ac} \sup_{t\in B(u,\varepsilon)} \mathsf{E}(\sup_{w\in B(u,\varepsilon)} |f'_u(t,w,V_1)|^{\alpha(w)}),$$

and for $j=3$,

$$\begin{aligned}
\sum_{n=2}^{\infty} \mathsf{P}(i^*=n) &\leq K|v-u|^{ac} \sup_{t\in B(u,\varepsilon)} \mathsf{E}(\sup_{w\in B(u,\varepsilon)} |f(t,w,V_1)\log|f(t,w,V_1)||^{\alpha(w)}) \\
&+ \sum_{i\geq 2} \mathsf{P}(A > |f(v,w_i,V_i)|^{\alpha(w_i)} > |v-u|^{-a\alpha(w_i)} \frac{Ki}{\log(i)^d}).
\end{aligned}$$

We have shown previously that the second term in the sum on the right hand side of the above inequality is bounded from above by $K|v-u|^{ac}$. Finally,

$$\lim_{v\to u} \mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j \mathbf{1}_{|Z_i^j|>|v-u|^{-a}}| > \frac{|v-u|^{\eta-1}}{2}\right) = 0.$$

Fourth step : verification of (I.12) for $\sum_{i=1}^{\infty} Y_i^j, j=1,2,3$.

We consider now $\mathsf{P}(|\sum_{i=1}^{\infty} Y_i^j| \geq |v-u|^{\eta-1})$.

Let $i^* = \inf\{n \geq 1 : i \geq n, |Y_i^j| \leq 1 \text{ and } \frac{1}{2} \leq \frac{\Gamma_i}{i} \leq 2\}$. Since $\sum\limits_{i\geq 1} \mathsf{P}(\{|Y_i^j| > 1\} \cup \{\Gamma_i < \frac{i}{2}\} \cup \{\Gamma_i > 2i\}) < +\infty$, the Borel-Cantelli lemma yields $\mathsf{P}(i^* = +\infty) = 0$. As a consequence,

$$\mathsf{P}(|\sum_{i=1}^{\infty} Y_i^j| \geq |v-u|^{\eta-1}) = \sum_{n\geq 1} \mathsf{P}\left(\{|\sum_{i=1}^{\infty} Y_i^j| \geq |v-u|^{\eta-1}\} \cap \{i^* = n\}\right).$$

Let $b_n(v) = \mathsf{P}\left(\{|\sum\limits_{i=1}^{\infty} Y_i^j| \geq |v-u|^{\eta-1}\} \cap \{i^* = n\}\right)$. Our strategy is the following : we show that, for each fixed n, $b_n(v)$ tends to 0 when v tends to u. Then we prove that there exists a summable sequence $(c_n)_n$ such that, for all n and all v, $b_n(v) \leq c_n$. We conclude using the dominated convergence theorem that $\sum_{n\geq 1} b_n(v)$ tends to 0 when v tends to u.

For all $n \geq 1$,

$$b_n(v) \leq \mathsf{P}(|\sum_{i=1}^{n-1} Y_i^j| \geq \frac{|v-u|^{\eta-1}}{2}) + \mathsf{P}(|\sum_{i=n}^{\infty} Y_i^j \mathbf{1}_{\{|Y_i^j|\leq 1\}\cap\{\frac{1}{2}\leq\frac{\Gamma_i}{i}\leq 2\}}| \geq \frac{|v-u|^{\eta-1}}{2}).$$

For $n \geq 2$, consider $\mathsf{P}(|\sum\limits_{i=1}^{n-1} Y_i^j| \geq \frac{|v-u|^{\eta-1}}{2})$.

$$\mathsf{P}(|\sum_{i=1}^{n-1} Y_i^j| \geq \frac{|v-u|^{\eta-1}}{2}) \leq \sum_{i=1}^{n-1} \mathsf{P}(|Y_i^j| \geq \frac{|v-u|^{\eta-1}}{2(n-1)}).$$

Let $p \in (0, \frac{c}{d})$. With K a positive constant that may change from line to line and depend on n but not on v, we have, for $j = 1$:

$$\begin{aligned}
\mathsf{P}\left(|Y_i^1| \geq \frac{|v-u|^{\eta-1}}{2(n-1)}\right) &\leq \mathsf{P}\left(\sup_{w\in B(u,\varepsilon)} |f(v,w,V_i)|^{\alpha(w)} |(\frac{\Gamma_i}{i})^{-1/\alpha(x_i)} - 1|^{\alpha(x_i)} \geq \frac{i|v-u|^{\alpha(x_i)(\eta-1)}}{(2(n-1)a'(x_i))^{\alpha(x_i)}}\right) \\
&\leq \mathsf{P}\left(\sup_{w\in B(u,\varepsilon)} |f(v,w,V_i)|^{\alpha(w)} |(\frac{\Gamma_i}{i})^{-1/\alpha(x_i)} - 1|^{\alpha(x_i)} \geq K|v-u|^{c(\eta-1)}\right) \\
&\leq \mathsf{P}\left((\sup_{w\in B(u,\varepsilon)} |f(v,w,V_i)|^{\alpha(w)})^p |(\frac{\Gamma_i}{i})^{-1/\alpha(x_i)} - 1|^{p\alpha(x_i)} \geq K|v-u|^{pc(\eta-1)}\right).
\end{aligned}$$

We use the following inequalities :

$$\begin{aligned}|(\frac{\Gamma_i}{i})^{-1/\alpha(x_i)}-1|^{p\alpha(x_i)} &= |(\frac{\Gamma_i}{i})^{-1/\alpha(x_i)}-1|^{p\alpha(x_i)}\mathbf{1}_{\Gamma_i>i}+|(\frac{\Gamma_i}{i})^{-1/\alpha(x_i)}-1|^{p\alpha(x_i)}\mathbf{1}_{\Gamma_i<i}\\ &\leq 1+|(\frac{\Gamma_i}{i})^{-1/\alpha(x_i)}-1|^{p\alpha(x_i)}\mathbf{1}_{\Gamma_i<i}\\ &\leq 1+|(\frac{\Gamma_i}{i})^{-1/c}-1|^{p\alpha(x_i)}\mathbf{1}_{\Gamma_i<i}\\ &\leq 1+|(\frac{\Gamma_i}{i})^{-1/c}-1|^{pc}+|(\frac{\Gamma_i}{i})^{-1/c}-1|^{pd},\end{aligned}$$

and obtain

$$\begin{aligned}\mathsf{P}\left(|Y_i^1|\geq\frac{|v-u|^{\eta-1}}{2(n-1)}\right) &\leq \mathsf{P}\left((\sup_{w\in B(u,\varepsilon)}|f(v,w,V_i)|^{\alpha(w)})^p\geq\frac{K}{3}|v-u|^{pc(\eta-1)}\right)\\ &+\mathsf{P}\left((\sup_{w\in B(u,\varepsilon)}|f(v,w,V_i)|^{\alpha(w)})^p|(\frac{\Gamma_i}{i})^{-1/c}-1|^{pc}\geq\frac{K}{3}|v-u|^{pc(\eta-1)}\right)\\ &+\mathsf{P}\left((\sup_{w\in B(u,\varepsilon)}|f(v,w,V_i)|^{\alpha(w)})^p|(\frac{\Gamma_i}{i})^{-1/c}-1|^{pd}\geq\frac{K}{3}|v-u|^{pc(\eta-1)}\right).\end{aligned}$$

Since $p<\frac{c}{d}<1$, $\mathsf{E}(|(\frac{\Gamma_i}{i})^{-1/c}-1|^{pc})<+\infty$ and $\mathsf{E}(|(\frac{\Gamma_i}{i})^{-1/c}-1|^{pd})<+\infty$ (this is easily verified by computing these expectations using the density of Γ_i). Using the independence of V_i and Γ_i and Markov inequality,

$$\begin{aligned}\mathsf{P}\left(|Y_i^1|\geq\frac{|v-u|^{\eta-1}}{2(n-1)}\right) &\leq K|v-u|^{pc(1-\eta)}\mathsf{E}\left((\sup_{w\in B(u,\varepsilon)}|f(v,w,V_i)|^{\alpha(w)})^p\right)\\ &\leq K|v-u|^{pc(1-\eta)},\end{aligned}$$

and

$$\lim_{v\to u}\mathsf{P}\left(|Y_i^1|\geq\frac{|v-u|^{\eta-1}}{2(n-1)}\right)=0.$$

Since the conditions required on (a',f) are also satisfied by (a,f'_u),

$$\lim_{v\to u}\mathsf{P}\left(|Y_i^2|\geq\frac{|v-u|^{\eta-1}}{2(n-1)}\right)=0.$$

We consider now the case $j=3$. When $i=1$:

$$\begin{aligned}\mathsf{P}\left(|Y_1^3|\geq\frac{|v-u|^{\eta-1}}{2(n-1)}\right) &= \mathsf{P}\left(|\log(\Gamma_1)\Gamma_1^{-1/\alpha(x_1)}f(v,x_1,V_1)|\geq\frac{\alpha(x_1)^2}{2|a(x_1)\alpha'(x_1)|(n-1)}|v-u|^{\eta-1}\right)\\ &\leq K|v-u|^{pc(1-\eta)}\mathsf{E}\left((\frac{|\log(\Gamma_1)|^c+|\log(\Gamma_1)|^d}{\Gamma_1})^p\right),\end{aligned}$$

(K depends on n but not on v). Since $p<1$ and α bounded, $\mathsf{E}\left(\left(\frac{|\log(\Gamma_1)|^c+|\log(\Gamma_1)|^d}{\Gamma_1}\right)^p\right)<+\infty$, and

$$\lim_{v\to u}\mathsf{P}\left(|Y_1^3|\geq\frac{|v-u|^{\eta-1}}{2(n-1)}\right)=0.$$

For $i\geq 2$,

$$\mathsf{P}\left(|Y_i^3|\geq\frac{|v-u|^{\eta-1}}{2(n-1)}\right) = \mathsf{P}\left(\left|\left(\frac{\log(\Gamma_i)}{\log(i)}(\frac{\Gamma_i}{i})^{-1/\alpha(x_i)}-1\right)f(v,x_i,V_i)\right|\geq\frac{\alpha(x_i)^2 i^{1/\alpha(x_i)}|v-u|^{\eta-1}}{\log(i)2|a(x_i)\alpha'(x_i)|(n-1)}\right).$$

One has :

$$\begin{aligned}|\frac{\log(\Gamma_i)}{\log(i)}(\frac{\Gamma_i}{i})^{-1/\alpha(x_i)}-1|^{\alpha(x_i)p} &\leq |\frac{\log(\Gamma_i)}{\log(i)}(\frac{\Gamma_i}{i})^{-1/c}-1|^{cp}+|\frac{\log(\Gamma_i)}{\log(i)}(\frac{\Gamma_i}{i})^{-1/c}-1|^{dp}+\\ &\quad |\frac{\log(\Gamma_i)}{\log(i)}(\frac{\Gamma_i}{i})^{-1/d}-1|^{cp}+|\frac{\log(\Gamma_i)}{\log(i)}(\frac{\Gamma_i}{i})^{-1/d}-1|^{dp}.\end{aligned}$$

Since $p\in(0,\frac{c}{d})$, the four terms in the right hand side of the above inequality have finite expectation (use again the density of Γ_i). Reasoning as in the case of Y_1^3, one gets :

$$\lim_{v\to u}\mathsf{P}\left(|Y_i^3|\geq\frac{|v-u|^{\eta-1}}{2(n-1)}\right)=0.$$

Finally, we have, for $j\in\{1,2,3\}$,

$$\lim_{v\to u}\mathsf{P}(\{|\sum_{i=1}^{n-1}Y_i^j|\geq\frac{|v-u|^{\eta-1}}{2}\})=0.$$

Let us now consider, for $n\geq 1$, $\mathsf{P}\left(\{|\sum_{i=n}^{\infty}Y_i^j\mathbf{1}_{\{|Y_i^j|\leq 1\}\cap\{\frac12\leq\frac{\Gamma_i}{i}\leq 2\}}|\geq\frac{|v-u|^{\eta-1}}{2}\}\right)$:

$$\begin{aligned}\mathsf{P}\left(\{|\sum_{i=n}^{\infty}Y_i^j\mathbf{1}_{\{|Y_i^j|\leq 1\}\cap\{\frac12\leq\frac{\Gamma_i}{i}\leq 2\}}|\geq\frac{|v-u|^{\eta-1}}{2}\}\right) &\leq 2|v-u|^{1-\eta}\mathsf{E}\left[|\sum_{i=n}^{\infty}Y_i^j\mathbf{1}_{\{|Y_i^j|\leq 1\}\cap\{\frac12\leq\frac{\Gamma_i}{i}\leq 2\}}|\right]\\ &\leq 2|v-u|^{1-\eta}\sum_{i=1}^{\infty}\mathsf{E}|Y_i^j|\mathbf{1}_{\{|Y_i^j|\leq 1\}\cap\{\frac12\leq\frac{\Gamma_i}{i}\leq 2\}}\\ &\leq K|v-u|^{1-\eta}\end{aligned}$$

(recall that the constants K used in bounding the series $\mathsf{E}(|Y_i^j|\mathbf{1}_{\{|Y_i^j|\leq 1\}\cap\{\frac12\leq\frac{\Gamma_i}{i}\leq 2\}})$ do not depend on v). Thus $b_n(v)\to 0$ when $v\to u$ for each n.

In view of using the dominated convergence theorem, we compute (recall that $B_i=\{\frac12\leq\frac{\Gamma_i}{i}\leq 2\}$) :

$$\begin{aligned}b_n(v) &\leq \mathsf{P}(\{i^*=n\})\\ &\leq \mathsf{P}(\{|Y_{n-1}^j|>1\}\cup\overline{B_{n-1}})\\ &\leq \mathsf{P}(\{|Y_{n-1}^j|>1\}\cap B_{n-1})+\mathsf{P}(\frac{\Gamma_{n-1}}{n-1}<\frac12)+\mathsf{P}(\frac{\Gamma_{n-1}}{n-1}>2).\end{aligned}$$

For $j = 1$ and $d \geq 1$,

$$\mathsf{P}(\{|Y^1_{n-1}| > 1\} \cap B_{n-1}) \leq \frac{K}{(n-1)^{1/\eta + c/2\eta}} (\sup_{t \in B(u,\varepsilon)} \mathsf{E}(\sup_{w \in B(u,\varepsilon)} |f(t,w,V_1)|^{\alpha(w)}))^{1/\eta}$$

and if $d < 1$,

$$\mathsf{P}(\{|Y^1_{n-1}| > 1\} \cap B_{n-1}) \leq \frac{K}{(n-1)^{1+c/2}} (\sup_{t \in B(u,\varepsilon)} \mathsf{E}(\sup_{w \in B(u,\varepsilon)} |f(t,w,V_1)|^{\alpha(w)})).$$

The same conclusion holds for $j = 2$, while, for $j = 3$,

$$\mathsf{P}(\{|Y^3_{n-1}| > 1\} \cap B_{n-1}) \leq K \frac{(\log(n-1))^d}{(n-1)^{1+c/2}} \mathbf{1}_{d<1} + K \frac{(\log(n-1))^{d/\eta}}{(n-1)^{1/\eta + c/2\eta}} \mathbf{1}_{d \geq 1}.$$

This finishes the proof ■

III.2 A Ferguson - Klass - LePage series representation of localisable processes in the σ-finite measure space case

When the space E has infinite measure, one cannot use the representation above, since it is no longer possible to renormalize by $m(E)$. This is a major drawback, since typical applications we have in mind deal with processes defined on the real line, *i.e.* $E = \mathbb{R}$ and m is the Lebesgue measure. However, in the σ-finite case, one may always perform a change of measure that allows to reduce to the finite case, as explained in [49], Proposition 3.11.3 (for specific examples of changes of measure, see Section III.3). In terms of localisability, this merely translates into adding a natural condition involving both the kernel and the change of measure :

Theorem III.5. *Let $(E, \mathcal{E}, m)$ be a σ-finite measure space. Let $r : E \to \mathbb{R}_+$ be such that $\hat{m}(dx) = \frac{1}{r(x)} m(dx)$ is a probability measure. Let α be a C^1 function defined on $\mathbb{R}$ and ranging in $(0,2)$. Let b be a C^1 function defined on $\mathbb{R}$. Let $f(t,u,.)$ be a family of functions such that, for all $(t,u) \in \mathbb{R}^2$, $f(t,u,.) \in \mathcal{F}_{\alpha(u)}(E, \mathcal{E}, m)$. Let $(\Gamma_i)_{i \geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i \geq 1}$ be a sequence of i.i.d. random variables with distribution $\hat{m}$ on E, and $(\gamma_i)_{i \geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Assume finally that the three sequences $(\Gamma_i)_{i \geq 1}$, $(V_i)_{i \geq 1}$, and $(\gamma_i)_{i \geq 1}$ are independent. Consider the following random field :*

$$X(t,u) = b(u) C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} r(V_i)^{1/\alpha(u)} f(t,u,V_i), \tag{III.6}$$

where $C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin(x) dx\right)^{-1}$. Assume that $X(t,u)$ (as a process in t) is localisable at u with exponent $h \in (0,1)$ and local form $X'_u(t,u)$. Assume in addition that :

- *(Cs1) The family of functions $v \to f(t,v,x)$ is differentiable for all (v,t) in a neighbourhood of u and almost all x in E. The derivatives of f with respect to v are denoted f'_v.*
- *(Cs3) There exists $\varepsilon > 0$ such that :*

$$\sup_{t \in B(u,\varepsilon)} \int_E \sup_{w \in B(u,\varepsilon)} (|f'_u(t,w,x)|^{\alpha(w)})\, m(dx) < \infty. \tag{III.7}$$

- *(Cs4) There exists $\varepsilon > 0$ such that :*

$$\sup_{t \in B(u,\varepsilon)} \int_E \sup_{w \in B(u,\varepsilon)} \left[|f(t,w,x) \log|f(t,w,x)||^{\alpha(w)}\right]\, m(dx) < \infty. \tag{III.8}$$

- *(Cs5) There exists $\varepsilon > 0$ such that :*

$$\sup_{t \in B(u,\varepsilon)} \int_E \sup_{w \in B(u,\varepsilon)} \left[|f(t,w,x) \log(r(x))|^{\alpha(w)}\right]\, m(dx) < \infty. \tag{III.9}$$

Then $Y(t) \equiv X(t,t)$ is localisable at u with exponent h and local form $Y'_u(t) = X'_u(t,u)$.

Remark : from (III.6), it may seem as though the process Y depends on the particular change of measure used, *i.e.* the choice of a specific r. However, this is not case. More precisely, Proposition III.12 below shows that the finite dimensional distributions of Y only depend on m.

Proof

We shall apply Theorem III.2 to the function $g(t,w,x) = r(x)^{1/\alpha(w)} f(t,w,x)$ on $(E, \mathcal{E}, \hat{m})$.

- By (Cs1), the family of functions $v \to f(t,v,x)$ is differentiable for all (v,t) in a neighbourhood of u and almost all x in E thus $v \to g(t,v,x)$ is differentiable too and (C1) holds.
- Choose $\varepsilon > 0$ such that (Cs4) and (Cs5) hold.

$$\int_{\mathbb{R}} \sup_{w \in B(u,\varepsilon)} \left[|g(t,w,x) \log|g(t,w,x)||^{\alpha(w)}\right]\, \hat{m}(dx)$$

$$= \int_{\mathbb{R}} r(x) \sup_{w \in B(u,\varepsilon)} \left[\left|f(t,w,x) \log|r(x)^{1/\alpha(w)} f(t,w,x)|\right|^{\alpha(w)}\right]\, \hat{m}(dx)$$

$$= \int_{\mathbb{R}} \sup_{w \in B(u,\varepsilon)} \left[\left|f(t,w,x) \log|r(x)^{1/\alpha(w)} f(t,w,x)|\right|^{\alpha(w)}\right]\, m(dx).$$

Expanding the logarithm above and using the inequality $|a+b|^\delta \leq \max(1, 2^{\delta-1})(|a|^\delta + |b|^\delta)$, valid for all real numbers a, b and all positive δ, one sees that (C4) holds.

- Choose $\varepsilon > 0$ such that (Cs3) and (Cs5) hold.

$$g'_u(t,w,x) = r(x)^{1/\alpha(w)} \left(f'_u(t,w,x) - \frac{\alpha'(w)}{\alpha^2(w)} \log(r(x)) f(t,w,x)\right)$$

and

$$\int_{\mathbb{R}} \sup_{w\in B(u,\varepsilon)} (|g'_u(t,w,x)|^{\alpha(w)})\ \hat{m}(dx)$$

$$= \int_{\mathbb{R}} \sup_{w\in B(u,\varepsilon)} \left[\left| f'_u(t,w,x) - \frac{\alpha'(w)}{\alpha^2(w)} \log(r(x)) f(t,w,x) \right|^{\alpha(w)} \right] m(dx).$$

The inequality $|a+b|^\delta \leq \max(1, 2^{\delta-1})(|a|^\delta + |b|^\delta)$ shows that (C3) holds.
Theorem III.2 allows to conclude ■

III.3 Examples of localisable processes

In this section, we apply the results above and obtain some localisable processes of interest. In particular, we consider "multistable versions" of several classical processes. Similar multistable extensions were considered in [16], to which the interested reader might refer for comparison.

We first recall some definitions. In the sequel, M will denote a symmetric α-stable ($0 < \alpha < 2$) random measure on $\mathbb{R}$ with control measure the Lebesgue measure $\mathcal{L}$. We will write

$$L_\alpha(t) := \int_0^t M(dz)$$

for α-stable Lévy motion.

The *log-fractional stable motion* is defined as

$$\Lambda_\alpha(t) = \int_{-\infty}^{\infty} (\log(|t-x|) - \log(|x|))\, M(dx) \quad (t \in \mathbb{R}).$$

This process is well-defined only for $\alpha \in (1,2]$ (the integrand does not belong to $\mathcal{F}_\alpha$ for $\alpha \leq 1$). Both Lévy motion and log-fractional stable motion are $1/\alpha$-self-similar with stationary increments.

The following process is called *linear fractional α-stable motion* :

$$L_{\alpha,H,b^+,b^-}(t) = \int_{-\infty}^{\infty} f_{\alpha,H}(b^+,b^-,t,x) M(dx)$$

where $t \in \mathbb{R}$, $H \in (0,1)$, $b^+, b^- \in \mathbb{R}$, and

$$\begin{aligned} f_{\alpha,H}(b^+,b^-,t,x) = b^+ &\left((t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha} \right) \\ &+ b^- \left((t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha} \right). \end{aligned}$$

L_{α,H,b^+,b^-} is again an sssi process. When $b^+ = b^- = 1$, this process is called well-balanced linear fractional α-stable motion and denoted $L_{\alpha,H}$.

Finally, for $\lambda > 0$, the stationary process

$$Y(t) = \int_t^{\infty} \exp(-\lambda(x-t)) M(dx) \quad (t \in \mathbb{R})$$

is called reverse Ornstein-Uhlenbeck process.

The localisability of Lévy motion, log-fractional stable motion and linear fractional α-stable motion simply stems from the fact that they are sssi. The localisability of the reverse Ornstein-Uhlenbeck process is proved in [15].

We will now define multistable versions of these processes.

For the multistable Lévy motion, we give two versions : one is fitted to the case where the time parameter varies in a compact interval $[0, T]$, and one where it spans $\mathbb{R}$.

Theorem III.6 (Symmetric multistable Lévy motion, compact case)**.** *Let $\alpha : [0, T] \to (1, 2)$ and $b : [0, T] \to]0, +\infty[$ be continuously differentiable. Let $(\Gamma_i)_{i\geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $\hat{m}(dx)$, the uniform distribution on $[0, T]$, and $(\gamma_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Assume finally that the three sequences $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, and $(\gamma_i)_{i\geq 1}$ are independent and define*

$$Y(t) = b(t) C_{\alpha(t)}^{1/\alpha(t)} T^{1/\alpha(t)} \sum_{i=1}^{+\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} 1_{[0,t]}(V_i) \quad (t \in [0, T]). \tag{III.10}$$

Then Y is $1/\alpha(u)$-localisable at any $u \in (0, T)$, with local form $Y'_u = b(u) L_{\alpha(u)}$.

The proof is a simple application of Theorem III.2, and is omitted.

Theorem III.7 (Symmetric multistable Lévy motion, non-compact case)**.** *Let $\alpha : \mathbb{R} \to (1, 2)$ and $b : \mathbb{R} \to]0, +\infty[$ be continuously differentiable. Let $(\Gamma_i)_{i\geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $\hat{m}(dx) = \sum_{j=1}^{+\infty} 2^{-j} \mathbf{1}_{[j-1,j[}(x)dx$ on $\mathbb{R}$, and $(\gamma_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Assume finally that the three sequences $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, and $(\gamma_i)_{i\geq 1}$ are independent and define*

$$Y(t) = b(t) C_{\alpha(t)}^{1/\alpha(t)} \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} 2^{j/\alpha(t)} 1_{[0,t] \cap [j-1,j[}(V_i) \quad (t \in \mathbb{R}_+). \tag{III.11}$$

Then Y is $1/\alpha(u)$-localisable at any $u \in \mathbb{R}_+$, with local form $Y'_u = b(u) L_{\alpha(u)}$.

Here and below for the other examples that we consider, we have chosen a specific function r in order to write an explicit representation of the multistable Levy motion in the non-compact case. In this example, $r(x) = \sum_{j=1}^{+\infty} 2^j \mathbf{1}_{[j-1,j[}(x)$. Section III.4 entails that the process does not depends on this specific choice, as long as r satisfies the conditions of Theorem III.5.

Proof

We apply Theorem III.5 with $m(dx) = dx$, $r(x) = \sum_{j=1}^{\infty} 2^j \mathbf{1}_{[j-1,j[}(x)$, $f(t, u, x) = \mathbf{1}_{[0,t]}(x)$ and the random field

$$X(t, u) = b(u) C_{\alpha(u)}^{1/\alpha(u)} \sum_{i,j=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} 2^{j/\alpha(u)} \mathbf{1}_{[0,t] \cap [j-1,j[}(V_i).$$

$X(.,u)$ is the symmetrical $\alpha(u)$-Lévy motion [49] and is thus $\frac{1}{\alpha(u)}$-localisable with local form $X'_u(.,u) = X(.,u)$.

- (Cs1) The family of functions $v \to f(t,v,x)$ is differentiable for all (v,t) in a neighbourhood of u and almost all x in E. The derivatives of f with respect to u vanish.
- (Cs3) $f'_u = 0$ so (Cs3) holds.
- (Cs4) $f(t,w,x)\log|f(t,w,x)| = 0$ so (Cs4) holds.
- (Cs5)

$$\begin{aligned} |f(t,w,x)\log(r(w))|^{\alpha(w)} &= \sum_{j=1}^{+\infty} j^{\alpha(w)} \log(2)^{\alpha(w)} \mathbf{1}_{[0,t]\cap[j-1,j[}(x) \\ &\leq \log(2)^c \sum_{j=1}^{+\infty} j^d \mathbf{1}_{[0,t]\cap[j-1,j[}(x) \end{aligned}$$

thus

$$\int_{\mathbb{R}} \sup_{w\in B(u,\varepsilon)} \left[|f(t,w,x)\log(r(x))|^{\alpha(w)} \right] dx \leq \log(2)^c \sum_{j=1}^{[t]+1} j^d$$

and (Cs5) holds ■

Theorem III.8 (Log-fractional multistable motion)**.** *Let $\alpha : \mathbb{R} \to (1,2)$ and $b : \mathbb{R} \to]0,+\infty[$ be continuously differentiable. Let $(\Gamma_i)_{i\geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $\hat{m}(dx) = \frac{3}{\pi^2}\sum_{j=1}^{+\infty} j^{-2}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)dx$ on $\mathbb{R}$, and $(\gamma_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Assume finally that the three sequences $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, and $(\gamma_i)_{i\geq 1}$ are independent and define*

$$Y(t) = b(t) C_{\alpha(t)}^{1/\alpha(t)} \sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} (\log|t-V_i| - \log|V_i|) \frac{\pi^{2/\alpha(t)}}{3^{1/\alpha(t)}} j^{2/\alpha(t)} \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(V_i) \quad (t\in\mathbb{R}). \tag{III.12}$$

Then Y is $1/\alpha(u)$-localisable at any $u \in \mathbb{R}$, with $Y'_u = b(u)\Lambda_{\alpha(u)}$.

Proof

We apply Theorem III.5 with $m(dx) = dx$, $r(x) = \frac{\pi^2}{3}\sum_{j=1}^{\infty} j^2 \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)$, $f(t,u,x) = \log(|t-x|) - \log(|x|)$ and the random field

$$X(t,u) = b(u) C_{\alpha(u)}^{1/\alpha(u)} \sum_{i,j=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} (\log|t-V_i| - \log|V_i|) \frac{\pi^{2/\alpha(u)}}{3^{1/\alpha(u)}} j^{2/\alpha(u)} \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(V_i).$$

$X(.,u)$ is the symmetrical $\alpha(u)$-Log-fractional motion. It is $\frac{1}{\alpha(u)}$-localisable with local form $X'_u(.,u) = b(u)\Lambda_{\alpha(u)}$ [49].

- (Cs1) The family of functions $v \to f(t,v,x)$ is differentiable for all (v,t) in a neighbourhood of u and almost all x in E. The derivatives of f with respect to u vanish.
- (Cs3) $f'_u = 0$ so (Cs3) holds.
- (Cs4)

$$\begin{aligned}|f(t,w,x)\log(|f(t,w,x)|)|^{\alpha(w)} &\leq |f(t,w,x)|^{\alpha(w)} + |f(t,w,x)\log(|f(t,w,x)|)|^{\alpha(w)}\mathbf{1}_{\{|f(t,w,x)|>e\}} \\ &+ |f(t,w,x)\log(|f(t,w,x)|)|^{\alpha(w)}\mathbf{1}_{\{|f(t,w,x)|<\frac{1}{e}\}}.\end{aligned}$$

We shall bound each of the three terms that are added up in the right hand side of the above inequality. For the first term,

$$|f(t,w,x)|^{\alpha(w)} \leq \sup_{w\in B(u,\varepsilon)} |f(t,w,x)|^{\alpha(w)}.$$

For the second term, fix $K>0$, $\delta>0$ such that $\forall x>e, |x\log(|x|)| \leq K|x|^{1+\delta}$.

$$|f(t,w,x)\log(|f(t,w,x)|)|^{\alpha(w)}\mathbf{1}_{\{|f(t,w,x)|>e\}} \leq K|f(t,w,x)|^{d(1+\delta)}.$$

For the third term, fix $K_2>0$, $\delta_2>0$ such that $c-\delta_2>1$ and

$$\forall |x| < \frac{1}{e}, |x|^c|\log|x||^d \leq K_2|x|^{c-\delta_2}.$$

This implies :

$$|f(t,w,x)\log(|f(t,w,x)|)|^{\alpha(w)}\mathbf{1}_{\{|f(t,w,x)|<\frac{1}{e}\}} \leq K_2|f(t,w,x)|^{c-\delta_2}.$$

We have then :

$$\begin{aligned}|f(t,w,x)\log(|f(t,w,x)|)|^{\alpha(w)} &\leq \sup_{w\in B(u,\varepsilon)} |f(t,w,x)|^{\alpha(w)} + K|f(t,w,x)|^{d(1+\delta)} \\ &+ K_2|f(t,w,x)|^{c-\delta_2}.\end{aligned}$$

For the first term,

$$\begin{aligned}|f(t,w,x)|^{\alpha(w)} &= |\log(|t-x|)-\log(|x|)|^{\alpha(w)} \\ &= |\log|1-\frac{t}{x}||^{\alpha(w)} \\ &\leq |\log|1-\frac{t}{x}||^d + |\log|1-\frac{t}{x}||^c.\end{aligned}$$

Then $\forall a>1$, $\exists K_a>0$ such that

$$\int_{\mathbb{R}} |f(t,w,x)|^a dx \leq K_a|t|. \tag{III.13}$$

The condition (Cs4) is then satisfied thanks to (III.13).

- (Cs5)

$$
\begin{aligned}
|f(t,w,x)\log(r(x))|^{\alpha(w)} &\leq K_1|f(t,w,x)\log(\frac{\pi^2}{3})|^{\alpha(w)} \\
&\quad + K_2\sum_{j=1}^{+\infty}|f(t,w,x)|^{\alpha(w)}(\log(j))^d\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x).
\end{aligned}
$$

For j large enough ($j > j^*$, where j^* does not depend on t,w,x), $|f(t,w,x)|^{\alpha(w)}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x) \leq K_5\frac{|t|^c}{|x|^c}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x))$. Thus

$$
|f(t,w,x)\log(r(x))|^{\alpha(w)} \leq K_6|f(t,w,x)|^{\alpha(w)} + K_7\sum_{j=j^*}^{+\infty}(\log(j))^d\frac{|t|^c}{|x|^c}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x).
$$

To conclude, note that

$$
\begin{aligned}
\int_{\mathbb{R}}(\log(j))^d\frac{1}{|x|^c}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)dx &= 2(\log(j))^d\int_{j-1}^{j}\frac{dx}{|x|^c} \\
&\sim 2\frac{(\log(j))^d}{j^c} \quad \blacksquare
\end{aligned}
$$

Theorem III.9 (Linear multistable multifractional motion)**.** *Let $b:\mathbb{R}\to]0,+\infty[$, $\alpha:\mathbb{R}\to(0,2)$ and $h:\mathbb{R}\to(0,1)$ be continuously differentiable. Let $(\Gamma_i)_{i\geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $\hat{m}(dx)=\frac{3}{\pi^2}\sum_{j=1}^{+\infty}j^{-2}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)dx$ on $\mathbb{R}$, and $(\gamma_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i=1)=P(\gamma_i=-1)=1/2$. Assume finally that the three sequences $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, and $(\gamma_i)_{i\geq 1}$ are independent and define for $t\in\mathbb{R}$*

$$
Y(t) = b(t)C_{\alpha(t)}^{1/\alpha(t)}\sum_{i,j=1}^{+\infty}\gamma_i\Gamma_i^{-1/\alpha(t)}(|t-V_i|^{h(t)-1/\alpha(t)}-|V_i|^{h(t)-1/\alpha(t)})(\frac{\pi^2j^2}{3})^{1/\alpha(t)}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(V_i). \tag{III.14}
$$

The process Y is $h(u)$-localisable at all $u\in\mathbb{R}$, with $Y'_u=b(u)L_{\alpha(u),h(u)}$ (the well balanced linear fractional stable motion).

Proof

We apply Theorem III.5 with $m(dx)=dx$, $r(x)=\frac{\pi^2}{3}\sum_{j=1}^{\infty}j^2\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)$, $f(t,u,x)=|t-x|^{h(u)-1/\alpha(u)}-|x|^{h(u)-1/\alpha(u)}$ and the random field

$$
X(t,u) = b(u)C_{\alpha(u)}^{1/\alpha(u)}(\frac{\pi^2j^2}{3})^{1/\alpha(u)}\sum_{i,j=1}^{\infty}\gamma_i\Gamma_i^{-1/\alpha(u)}(|t-V_i|^{h(u)-1/\alpha(u)}-|V_i|^{h(u)-1/\alpha(u)})\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(V_i).
$$

$X(.,u)$ is the $(\alpha(u),h(u))$-well balanced linear fractional stable motion and it is $\frac{1}{\alpha(u)}$-localisable with local form $X'_u(.,u)=b(u)L_{\alpha(u),h(u)}$ [16].

- (Cs1) The family of functions $v\to f(t,v,x)$ is differentiable for all (v,t) in a neighbourhood of u and almost all x in $\mathbb{R}$. The derivatives of f with respect to u read :

$$f'_u(t,w,x)=(h'(w)+\frac{\alpha'(w)}{\alpha^2(w)})\left[(\log|t-x|)|t-x|^{h(w)-1/\alpha(w)}-(\log|x|)|x|^{h(w)-1/\alpha(w)}\right].$$

- (Cs3) First we note (Cs2) the following condition :
There exists $\varepsilon>0$ such that :

$$\sup_{t\in B(u,\varepsilon)}\int_E\sup_{w\in B(u,\varepsilon)}(|f(t,w,x)|^{\alpha(w)})\,m(dx)<\infty. \tag{III.15}$$

In [16], it is shown that, given $u\in\mathbb{R}$, one may choose $\varepsilon>0$ small enough and numbers a,b,h_-,h_+ with $0<a<\alpha(w)<b<2$, $0<h_-<h(w)<h_+<1$ and $\frac{1}{a}-\frac{1}{b}<h_-<h_+<1-(\frac{1}{a}-\frac{1}{b})$ such that, for all t and w in $U:=(u-\varepsilon,u+\varepsilon)$ and all real x :

$$|f(t,w,x)|,|f'_u(t,w,x)|\le k_1(t,x) \tag{III.16}$$

where

$$k_1(t,x)=\begin{cases} c_1\max\{1,|t-x|^{h_--1/a}+|x|^{h_--1/a}\} & (|x|\le 1+2\max_{t\in U}|t|)\\ c_2|x|^{h_+-1/b-1} & (|x|>1+2\max_{t\in U}|t|)\end{cases} \tag{III.17}$$

for appropriately chosen constants c_1 and c_2. The conditions on a,b,h_-,h_+ entail that $\sup_{t\in U}\|k_1(t,\cdot)\|_{a,b}<\infty$, where $\|k_1(t,\cdot)\|_{a,b}$ is defined by (I.13), and (Cs2) hold for k_1. (Cs3) is obtained with (III.16).
- (Cs4)

$$\begin{aligned}|f(t,w,x)\log(|f(t,w,x)|)|^{\alpha(w)} &\le |f(t,w,x)|^{\alpha(w)}+|f(t,w,x)\log(|f(t,w,x)|)|^{\alpha(w)}\mathbf{1}_{\{|f(t,w,x)|>e\}}\\ &\quad+|f(t,w,x)\log(|f(t,w,x)|)|^{\alpha(w)}\mathbf{1}_{\{|f(t,w,x)|<\frac{1}{e}\}}.\end{aligned}$$

Since

$$|f(t,w,x)|\le k_1(t,x) \tag{III.18}$$

one gets

$$\begin{aligned}|f(t,w,x)\log(|f(t,w,x)|)|\mathbf{1}_{\{|f(t,w,x)|>e\}} &\le k_1(t,x)\log(k_1(t,x))\mathbf{1}_{\{|f(t,w,x)|>e\}}\\ &\le |k_1(t,x)\log(k_1(t,x))|\\ |f(t,w,x)\log(|f(t,w,x)|)|^{\alpha(w)}\mathbf{1}_{\{|f(t,w,x)|>e\}} &\le |k_1(t,x)\log(k_1(t,x))|^{\alpha(w)}.\end{aligned}$$

Fix $\eta > 0$ such that $1 < \eta < a + \frac{a}{b} - ah_+$ and $\lambda > 0$ such that $\frac{1}{\eta} < \lambda < 1$.

$$\begin{aligned} |f(t,w,x)\log(|f(t,w,x)|)|^{\alpha(w)}\mathbf{1}_{\{|f(t,w,x)|<\frac{1}{e}\}} &\leq K|f(t,w,x)|^{\lambda\alpha(w)} \\ &\leq K|k_1(t,x)|^{\lambda\alpha(w)} \end{aligned}$$

and thus, since k_1 satisfies the conditions (Cs2) and (Cs4), (Cs4) holds for f.

- (Cs5) For j large enough ($j > j^*$),

$$\begin{aligned} |f(t,w,x)\log(r(x))|^{\alpha(w)} &\leq K_1|k_1(t,x)|^{\alpha(w)} \\ &\quad + K_2\sum_{j=j^*}^{+\infty}|f(t,w,x)|^{\alpha(w)}(\log(j))^d\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x). \end{aligned}$$

$$|f(t,w,x)|^{\alpha(w)}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x) \leq K_3\frac{1}{|x|^{a(1+1/b-h_+)}}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x).$$

Thus

$$|f(t,w,x)\log(r(x))|^{\alpha(w)} \leq K_1|k_1(t,x)|^{\alpha(w)} + K_4\sum_{j=j^*}^{+\infty}\frac{(\log(j))^d}{|x|^{a(1+1/b-h_+)}}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x).$$

To conclude, note that

$$\begin{aligned} \int_{\mathbb{R}}\frac{(\log(j))^d}{|x|^{a(1+1/b-h_+)}}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)dx &= 2(\log(j))^d\int_{j-1}^{j}\frac{dx}{|x|^{a(1+1/b-h_+)}} \\ &\sim 2\frac{(\log(j))^d}{j^{a(1+1/b-h_+)}} \quad \blacksquare \end{aligned}$$

Theorem III.10 (Multistable reverse Ornstein-Uhlenbeck process)**.** *Let $\lambda > 0$, $\alpha : \mathbb{R} \to (1,2)$ and $b : \mathbb{R} \to]0,+\infty[$ be continuously differentiable. Let $(\Gamma_i)_{i\geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $\hat{m}(dx) = \sum_{j=1}^{+\infty}2^{-j-1}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)dx$ on $\mathbb{R}$, and $(\gamma_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Assume finally that the three sequences $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, and $(\gamma_i)_{i\geq 1}$ are independent and define*

$$Y(t) = b(t)C_{\alpha(t)}^{1/\alpha(t)}\sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty}\gamma_i\Gamma_i^{-1/\alpha(t)}2^{(j+1)/\alpha(t)}e^{-\lambda(V_i-t)}\mathbf{1}_{[t,+\infty)\cap([-j,-j+1)\cup[j-1,j))}(V_i) \quad (t \in \mathbb{R}). \tag{III.19}$$

Then Y is $1/\alpha(u)$-localisable at any $u \in \mathbb{R}$, with local form $Y'_u = b(u)L_{\alpha(u)}$.

Proof
We apply Theorem III.5 with $m(dx) = dx$, $r(x) = \sum_{j=1}^{\infty} 2^{j+1}\mathbf{1}_{[-j,-j+1)\cup[j-1,j)}(x)$, $f(t,u,x) = e^{-\lambda(x-t)}\mathbf{1}_{[t,+\infty)}(x)$ and the random field

$$X(t,u) = b(u)C_{\alpha(u)}^{1/\alpha(u)}\sum_{i,j=1}^{\infty}\gamma_i\Gamma_i^{-1/\alpha(u)}2^{(j+1)/\alpha(u)}e^{-\lambda(V_i-t)}\mathbf{1}_{[t,+\infty)\cap([-j,-j+1)\cup[j-1,j))}(V_i).$$

$X(.,u)$ is the symmetrical $\alpha(u)$-reverse Ornstein-Uhlenbeck process and is $\frac{1}{\alpha(u)}$-localisable with local form $X_u'(.,u) = b(u)L_{\alpha(u)}$ [15].

- (Cs1) The family of functions $v \to f(t,v,x)$ is differentiable for all (v,t) in a neighbourhood of u and almost all x in E. The derivatives of f with respect to u vanish.
- (Cs3) $f_u' = 0$ so (Cs3) holds.
- (Cs4)

$$\begin{aligned}|f(t,w,x)\log(|f(t,w,x)|)|^{\alpha(w)} &= \lambda^{\alpha(w)}(x-t)^{\alpha(w)}e^{-\lambda\alpha(w)(x-t)}\mathbf{1}_{[t,+\infty[}(x)\\ &\le \max(1,\lambda^d)\max(1,(x-t)^d)e^{-\lambda c(x-t)}\mathbf{1}_{[t,+\infty[}(x)\end{aligned}$$

as a consequence

$$\begin{aligned}\int_{\mathbb{R}}\sup_{w\in B(u,\varepsilon)}\left[|f(t,w,x)\log(|f(t,w,x)|)|^{\alpha(w)}\right]dx &\le \max(1,\lambda^d)\left(\int_0^1 e^{-\lambda cu}\,du + \int_0^1 u^d e^{-\lambda cu}\,du\right)\\ &< +\infty.\end{aligned}$$

- (Cs5)

$$|f(t,w,x)\log(r(x))|^{\alpha(w)} = \sum_{j=1}^{+\infty}(j+1)^{\alpha(w)}\log(2)^{\alpha(w)}e^{-\lambda\alpha(w)(x-t)}\mathbf{1}_{[t,+\infty[\cap([-j,-j+1[\cup[j-1,j[)}(x).$$

Fix j^* large enough such that for all $j > j^*$, $\mathbf{1}_{[t,+\infty[\cap([-j,-j+1[\cup[j-1,j[)}(x) = \mathbf{1}_{[j-1,j[}(x)$. Then

$$\int_{\mathbb{R}}\sup_{w\in B(u,\varepsilon)}\left[|f(t,w,x)\log(r(x))|^{\alpha(w)}\right]dx \le \sum_{j=1}^{j^*}\frac{(j+1)^d\log(2)^c}{\lambda c}$$

$$+\sum_{j=j^*+1}^{+\infty}(j+1)^d\log(2)^c\int_{j-1}^{j}e^{-\lambda c(x-t)}\,dx$$

$$\le \sum_{j=1}^{j^*}\frac{(j+1)^d\log(2)^c}{\lambda c} + \log(2)^c e^{\lambda ct}(e^{\lambda c}-1)\sum_{j=j^*+1}^{+\infty}\frac{(j+1)^d e^{-\lambda cj}}{\lambda c} \quad \blacksquare$$

We provide now an example of a localisable process where the kernel f is not satisfying the criteria of [16]. We can state that the process is localisable thanks to Theorem III.2, but we cannot use Theorem 5.2 of [16] anymore. Indeed, for all open set $U \subset (0, 1/e)$, we will have $\sup_{t\in U} \|f(t,u,.)\|_{c,d} = +\infty$ where $c = \inf_{v\in U} \alpha(v)$, $d = \sup_{v\in U} \alpha(v)$ and $\|f(t,u,.)\|_{c,d}$ defined by (I.13).

Theorem III.11. *Let $\alpha : U := (0, \frac{1}{2e}) \to \mathbb{R}$, $\alpha(t) = 1 + t$. Let $(\Gamma_i)_{i\geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $\hat{m}(dx)$, the uniform distribution on U, and $(\gamma_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Assume finally that the three sequences $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, and $(\gamma_i)_{i\geq 1}$ are independent and define*

$$Y(t) = C_{\alpha(t)}^{1/\alpha(t)} \frac{1}{(2e)^{1/\alpha(t)}} \sum_{i=1}^{+\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} \frac{1}{V_i^{1/\alpha(t)} |\ln V_i|^{4/\alpha(t)}} 1_{[0,t]}(V_i) \quad (t \in U). \tag{III.20}$$

Then Y is $1/\alpha(u)$-localisable at any $u \in U$, with local form $Y'_u = \frac{1}{u^{1/\alpha(u)}|\ln u|^{4/\alpha(u)}} L_{\alpha(u)}$.

Proof

We apply Theorem III.2 with $m(dx) = dx$, $f(t,u,x) = \frac{1}{x^{1/\alpha(u)}|\ln x|^{4/\alpha(u)}} \mathbf{1}_{[0,t]}(x)$ and the random field

$$X(t,u) = C_{\alpha(u)}^{1/\alpha(u)} \frac{1}{(2e)^{1/\alpha(u)}} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} \frac{1}{V_i^{1/\alpha(u)} |\ln V_i|^{4/\alpha(u)}} \mathbf{1}_{[0,t]}(V_i).$$

$X(.,u)$ is $\frac{1}{\alpha(u)}$-localisable with local form $X'_u(.,u) = \frac{1}{u^{1/\alpha(u)}|\ln u|^{4/\alpha(u)}} L_{\alpha(u)}$.

- (C1) The family of functions $v \to f(t,v,x)$ is differentiable for all (v,t) in a neighbourhood of u and almost all x in E. The derivatives of f with respect to v is
$$\left(\frac{\alpha'(v)\ln x}{\alpha^2(v)} - \frac{16\alpha'(v)\ln(|\ln x|)}{\alpha^3(v)} \right) \frac{1}{x^{1/\alpha(v)}|\ln x|^{4/\alpha(v)}} \mathbf{1}_{[0,t]}(x).$$
- (C3) There exists a positive constant K such that
$$\sup_{w\in B(u,\varepsilon)} |f'_v(t,w,x)|^{\alpha(w)} \leq \frac{K}{x|\ln x|^4} \left(|\ln x|^2 + |\ln(|\ln x|)| + |\ln(|\ln x|)|^2 \right) \mathbf{1}_{[0,t]}(x),$$
so (C3) holds.
- (C4) We compute
$$|f(t,w,x)\log|f(t,w,x)||^{\alpha(w)} = \frac{1}{x|\ln x|^4} \left(-\frac{\ln x}{\alpha(w)} - \frac{4\ln|\ln x|}{\alpha(w)} \right)^{\alpha(w)} \mathbf{1}_{[0,t]}(x),$$
so there exists a positive constant K such that
$$\sup_{w\in B(u,\varepsilon)} |f(t,w,x)\log|f(t,w,x)||^{\alpha(w)} \leq \frac{K}{x|\ln x|^4} \left(|\ln x|^2 + |\ln(|\ln x|)| + |\ln(|\ln x|)|^2 \right) \mathbf{1}_{[0,t]}(x),$$

and (C4) holds ■

III.4 Finite dimensional distributions

In this section, we compute the finite dimensional distributions of the family of processes defined in Theorem III.5, and compare the results with the ones in [16].

Proposition III.12. With notations as above, let $\{X(t,u), t, u \in \mathbb{R}\}$ be as in $(III.6)$ and $Y(t) \equiv X(t,t)$. The finite dimensional distributions of the process Y are equal to

$$\mathsf{E}\left(e^{i\sum_{j=1}^{m}\theta_j Y(t_j)}\right) = \exp\left(-2\int_E\int_0^{+\infty}\sin^2\Big(\sum_{j=1}^{m}\theta_j b(t_j)\frac{C_{\alpha(t_j)}^{1/\alpha(t_j)}}{2y^{1/\alpha(t_j)}}f(t_j,t_j,x)\Big)\,dy\,m(dx)\right)$$

for $m \in \mathbb{N}, \boldsymbol{\theta} = (\theta_1, \dots, \theta_m) \in \mathbb{R}^m, \mathbf{t} = (t_1, \dots, t_m) \in \mathbb{R}^m$.

Proof

Let $m \in \mathbb{N}$ and write $\phi_t(\theta) = \mathsf{E}\left(e^{i\sum_{j=1}^{m}\theta_j Y(t_j)}\right)$. We proceed as in [49], Theorem 1.4.2. Let $\{U_i\}_{i\in\mathbb{N}}$ be an i.i.d sequence of uniform random variables on $(0,1)$, independent of the sequences $\{\gamma_i\}$ and $\{V_i\}$, and $g(t,u,x) = b(u)C_{\alpha(u)}^{1/\alpha(u)}r(x)^{1/\alpha(u)}f(t,u,x)$. For all $n \in \mathbb{N}$,

$$\sum_{j=1}^{m}\theta_j n^{-1/\alpha(t_j)}\sum_{k=1}^{n}\gamma_k U_k^{-1/\alpha(t_j)}g(t_j,t_j,V_k) \stackrel{d}{=} \sum_{j=1}^{m}\theta_j\left(\frac{\Gamma_{n+1}}{n}\right)^{1/\alpha(t_j)}\sum_{k=1}^{n}\gamma_k\Gamma_k^{-1/\alpha(t_j)}g(t_j,t_j,V_k). \tag{III.21}$$

The right-hand side of (III.21) converges almost surely to $\sum_{j=1}^{m}\theta_j Y(t_j)$ when n tends to infinity and thus

$$\phi_t(\theta) = \lim_{n\to+\infty}\mathsf{E}\left(e^{i\sum_{j=1}^{m}\theta_j n^{-1/\alpha(t_j)}\sum_{k=1}^{n}\gamma_k U_k^{-1/\alpha(t_j)}g(t_j,t_j,V_k)}\right).$$

Set $\phi_t^n(\theta) = \mathsf{E}\left(e^{i\sum_{j=1}^{m}\theta_j n^{-1/\alpha(t_j)}\sum_{k=1}^{n}\gamma_k U_k^{-1/\alpha(t_j)}g(t_j,t_j,V_k)}\right)$. This function may be written as :

$$\phi_t^n(\theta) = \mathsf{E}\left(\prod_{k=1}^{n}e^{i\gamma_k\sum_{j=1}^{m}\theta_j n^{-1/\alpha(t_j)}U_k^{-1/\alpha(t_j)}g(t_j,t_j,V_k)}\right).$$

All the sequences $\{\gamma_k\}$, $\{U_k\}$, $\{V_k\}$ are i.i.d. As a consequence,

$$\phi_t^n(\theta) = \left(\mathsf{E}\left(e^{i\gamma_1\sum_{j=1}^{m}\theta_j n^{-1/\alpha(t_j)}U_1^{-1/\alpha(t_j)}g(t_j,t_j,V_1)}\right)\right)^n.$$

We compute now the expectation using conditioning and independence of the sequences $\{\gamma_k\}$, $\{U_k\}$ and $\{V_k\}$.

$$\begin{aligned}
\mathsf{E}\left(e^{i\gamma_1 \sum_{j=1}^{m} \theta_j n^{-1/\alpha(t_j)} U_1^{-1/\alpha(t_j)} g(t_j,t_j,V_1)}\right) &= \mathsf{E}\left(\mathsf{E}\left(e^{i\gamma_1 \sum_{j=1}^{m} \theta_j n^{-1/\alpha(t_j)} U_1^{-1/\alpha(t_j)} g(t_j,t_j,V_1)} |U_1, V_1\right)\right) \\
&= \mathsf{E}\left(\cos(\sum_{j=1}^{m} \theta_j n^{-1/\alpha(t_j)} U_1^{-1/\alpha(t_j)} g(t_j,t_j,V_1))\right) \\
&= \mathsf{E}\left(\frac{1}{n}\int_0^n \cos(\sum_{j=1}^{m} \theta_j y^{-1/\alpha(t_j)} g(t_j,t_j,V_1))\, dy\right) \\
&= 1 - \frac{2}{n}\int_0^n \mathsf{E}\left(\sin^2(\sum_{j=1}^{m} \frac{\theta_j}{2} y^{-1/\alpha(t_j)} g(t_j,t_j,V_1))\right) dy.
\end{aligned}$$

The function $\sin^2$ is positive and thus, when n tends to $+\infty$,

$$\int_0^n \mathsf{E}\left(\sin^2(\sum_{j=1}^{m} \frac{\theta_j}{2} y^{-1/\alpha(t_j)} g(t_j,t_j,V_1))\right) dy \to \int_0^{+\infty} \mathsf{E}\left(\sin^2(\sum_{j=1}^{m} \frac{\theta_j}{2} y^{-1/\alpha(t_j)} g(t_j,t_j,V_1))\right) dy.$$

To conclude, note that

$$\mathsf{E}\left(\sin^2(\sum_{j=1}^{m} \frac{\theta_j}{2} y^{-1/\alpha(t_j)} g(t_j,t_j,V_1))\right) = \int_E \sin^2(\sum_{j=1}^{m} \frac{\theta_j}{2} y^{-1/\alpha(t_j)} g(t_j,t_j,x))\, \hat{m}(dx) \quad \blacksquare$$

Comparing with Proposition 8.2, Theorems 9.3, 9.4, 9.5 and 9.6 in [16], it is easy to prove the following corollary, which shows that the approach based on the series representation and the one based on sums over Poisson processes yield essentially the same processes :

Corollary III.13. The linear multistable multifractional motion, multistable Lévy motion, log-fractional multistable motion and multistable reverse Ornstein-Uhlenbeck process defined in Section III.3 have the same finite dimensional distributions as the corresponding processes considered in [16].

III.5 Numerical experiments

We display in this section graphs of synthesized paths of some of the processes defined above. The idea is just to picture how multistability translates on the behaviour of random trajectories, and, in the case of linear multistable multifractional motion, to visualize the effect of both a varying H and a varying α, these two parameters corresponding to two different notions of irregularity. The synthesis method is described in [15]. Theoretical results concerning the convergence of this method will be presented elsewhere.

The two graphs on the first line of Figure III.1 ((a) and (b)) display multistable Lévy motions, with respectively α increasing linearly from 1.02 to 1.98 (shown in (d)) and α a sine function ranging in the same interval (shown in (c)). The graph (e) displays an Ornstein-Uhlenbeck multistable process with same sine α function. A linear multistable multifractional motion with linearly increasing α and H functions is shown in (f). H increases from 0.2 to 0.8 and α from 1.41 to 1.98 (these two functions are displayed on the right part of the bottom line). The graph in (g) is again a linear multistable multifractional motion, but with linearly increasing α and linearly decreasing H. H decreases from 0.8 to 0.2 and α increases from 1.41 to 1.98 (these two functions are displayed on the left part of the bottom line). Finally, a zoom on the second half of the process in (f) is shown, that allows to appreciate how the graph becomes smoother as H increases.

In all these graphs, one clearly sees how the variations of α translates in terms of the "intensity" of jumps. Additionally, in the case of linear multistable multifractional motions, the interplay between the smoothness governed by H and the stability function α indicate that such processes may prove useful in various applications such as finance or biomedical signal modeling. However, the graphs (f) and (g), observed in the range $\alpha(t)H(t) < 1$, are a little misleading. In that case, the linear multistable multifractional motion is in fact unbounded on every interval, which is not apparent on the plots.

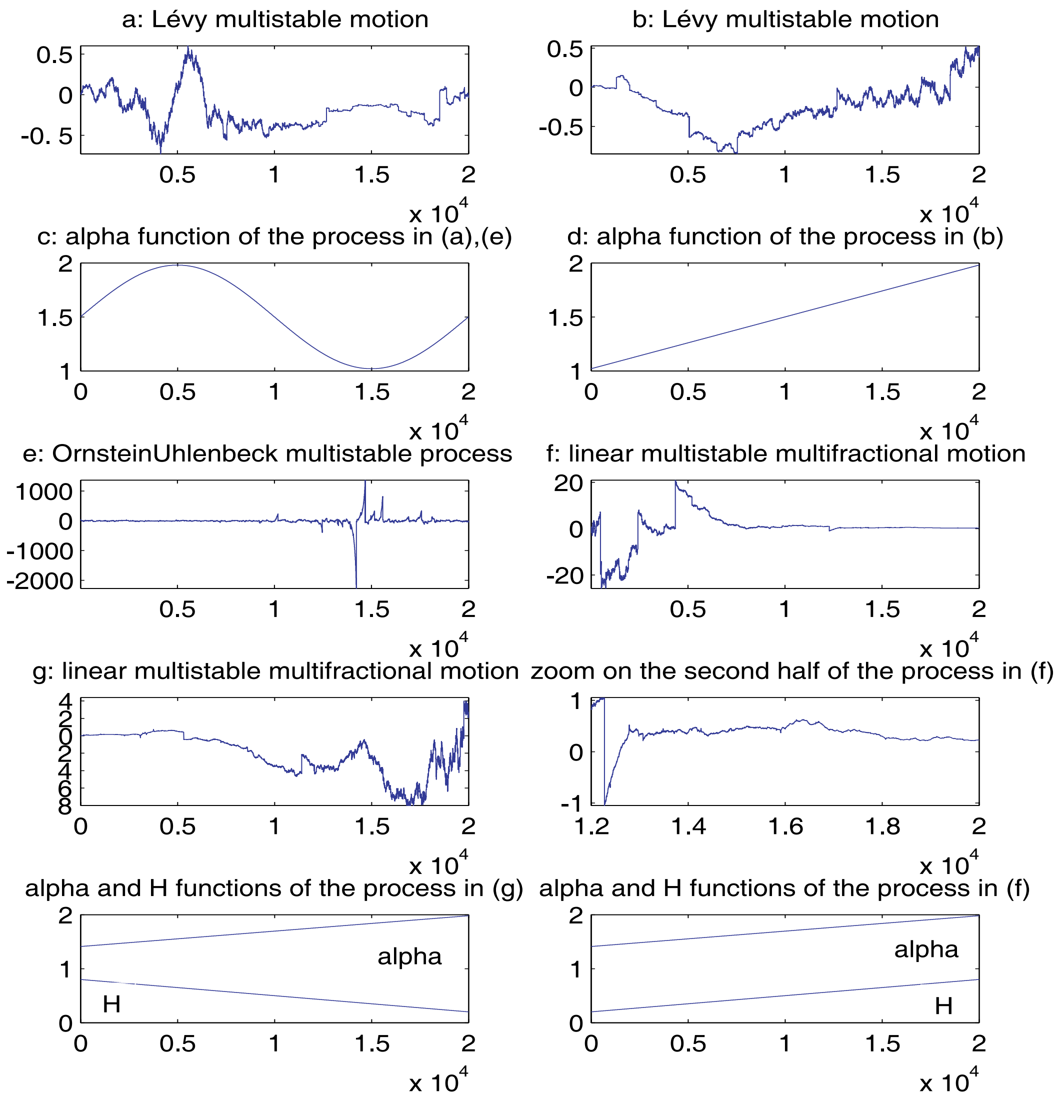


FIGURE III.1 – Paths of multistable processes. First line : Levy multistable motions with sine (a) and linear (b) α function. Second line : (c) α function for the process in (a), (d) α function for the process in (b). Third line : (e) multistable Ornstein-Uhlenbeck process with α function displayed in (c), and (f) linear multistable multifractional motion with linear increasing α and H functions. Fourth line : (g) linear multistable multifractional motion with linear increasing α function and linear decreasing H function, and zoom on the second part of the process in (f). Last line : α and H functions for the process in (g) (left), α and H functions for the process in (f) (right).

Chapitre IV

Moments des accroissements et exposants de Hölder des processus multifractionnaires multistables

Plan du chapitre

Abstract

In this work, we give further results on (multifractional) multistable processes related to their local structure. We show that, under certain conditions, the incremental moments display a scaling behaviour, and that the pointwise Hölder exponent is, as expected, related to the local stability index. We compute the precise value of the almost sure Hölder exponent in the case of the multistable Lévy motion, which turns out to reveal an interesting phenomenon.

The aim of this work is twofold :

1. We show that, for a large class of (multifractional) multistable processes, a precise estimate for the incremental moments holds. More precisely, we prove in Section IV.1.1 that there exists a natural scaling relation for $\mathsf{E}\left[|Y(t+\varepsilon)-Y(t)|^\eta\right]$ and ε small. This class includes (multifractional) multistable processes considered in [16,29], in particular Lévy multistable motions and linear multistable multifractional motions.
2. We then study the pointwise Hölder regularity of (multifractional) multistable processes. For the same class as above, we obtain an almost sure upper bound for this exponent. In the case of the Lévy multistable motion, we are able to compute its exact value. An interesting phenomenon occurs : when the functional parameter α is smooth, not surprisingly, the Hölder exponent is equal, at each point, almost surely, to the localisability index. However, when α is smaller than one and sufficiently irregular, the regularity of the process is governed by the one of α : their Hölder exponent coincide. Note that a uniform statement, *i.e.* a statement like "almost surely, at each point", cannot hold true in general. Indeed, it already fails for the case of a Lévy stable motion. The right frame in this respect is multifractal analysis, and results in this direction will be presented in a forthcoming work.

The remainder of this work is organized as follows. Our main results on incremental moments and upper bound for the pointwise Hölder exponents are described in Subsections IV.1.1 and IV.1.2. Subsection IV.1.3 applies these findings to the linear multistable multifractional motion. In Subsection IV.1.4, we state the result giving the exact value of the pointwise Hölder regularity of the Lévy multistable motion. In Section IV.2, we give intermediate results, some of which being of independent interest, which are used in the proofs of the main statements. Section IV.3 gathers technical results followed by the proofs of the statements related with the incremental moments and upper bounds on the exponents. Section IV.4 contains the computation of the exponent for the multistable Lévy motion. Finally, Section IV.5 gives a list of the various technical conditions on multistable processes required by our approach so that their incremental moments and Hölder exponents may be estimated.

IV.1 Main results

The two following theorems apply to a diagonal process Y defined from the field X given by (I.19) or (I.20). For convenience, the conditions required on X and the function f that appears in (I.19) and (I.20), denoted (C1), ..., (C15), are gathered in Section IV.5.

IV.1.1 Moments of multistable processes

Theorem IV.1. *Let $t\in\mathbb{R}$ and U be an open interval of $\mathbb{R}$ with $t\in U$. Let $\eta\in(0,c)$. Suppose that f satisfies (C1), (C2), (C3) (or (C1), (Cs2), (Cs3), (Cs4) in the σ-finite case), and (C9), and that X verifies (C5) at t. Then, when ε tends to 0,*

$$\mathsf{E}\left[|Y(t+\varepsilon)-Y(t)|^\eta\right]\sim\varepsilon^{\eta h(t)}\mathsf{E}\left[|Y_t'(1)|^\eta\right].$$

Proof
See Section IV.3.

Remark : Under the conditions listed in the theorem, Theorems 3.3 and 4.5 of [29] imply that Y is $h(t)-$localisable at t.

IV.1.2 Pointwise Hölder exponent of multistable processes

Let $\mathcal{H}_t = \sup\{\gamma : \lim_{r\to 0} \frac{|Y(t+r)-Y(t)|}{|r|^\gamma} = 0\}$ denote the Hölder exponent of the (non-differentiable) process Y at t.

Theorem IV.2 (Upper bound)**.** *Suppose that there exists a function h defined on U such that (C6), (C7), (C8), (C10), (C11), (C12), (C13), (C14) and (C15) holds for some $t \in U$. Assuming (C1), (C2), (C3), (or (C1), (Cs2), (Cs3), (Cs4) in the σ-finite case), one has :*

$$\mathcal{H}_t \leq h(t).$$

Proof
See Section IV.3.

IV.1.3 Example : the linear multistable multifractional motion

In this section, we apply the results above to the "multistable version" of a classical process known as the linear stable multifractional motion, which is itself a extension of the linear stable fractional motion, defined as follows (in the sequel, M will always denote a symmetric α-stable ($0 < \alpha < 2$) random measure on $\mathbb{R}$ with control measure Lebesgue measure $\mathcal{L}$) :

$$L_{\alpha,H,b^+,b^-}(t) = \int_{-\infty}^{\infty} f_{\alpha,H}(b^+, b^-, t, x) M(dx)$$

where $t \in \mathbb{R}$, $H \in (0,1)$, $b^+, b^- \in \mathbb{R}$, and

$$\begin{aligned} f_{\alpha,H}(b^+, b^-, t, x) = b^+ &\left((t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}\right) \\ &+ b^- \left((t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha}\right). \end{aligned}$$

When $b^+ = b^- = 1$, this process is called well-balanced linear fractional α-stable motion and denoted $L_{\alpha,H}$.

The localisability of the linear fractional α-stable motion simply stems from the fact that it is $1/\alpha$-self-similar with stationary increments [14].

The multistable version of this processes was defined in [15, 16]. Its incremental moments and regularity are described by the following theorems :

Theorem IV.3 (Linear multistable multifractional motion)**.** *Let $\alpha : \mathbb{R} \to [c, d] \subset (0, 2)$ and $H : \mathbb{R} \to (0, 1)$ be continuously differentiable. Let $(\Gamma_i)_{i\geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with* *distribution* $\hat{m}(dx) = \frac{3}{\pi^2}\sum_{j=1}^{+\infty} j^{-2}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)dx$ *on* $\mathbb{R}$*, and* $(\gamma_i)_{i\geq 1}$ *be a sequence of i.i.d. random variables with distribution* $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$*. Assume finally that the three sequences* $(\Gamma_i)_{i\geq 1}$*,* $(V_i)_{i\geq 1}$*, and* $(\gamma_i)_{i\geq 1}$ *are independent and define*

$$X(t,u) = C_{\alpha(u)}^{1/\alpha(u)} \sum_{i,j=1}^{\infty} (\frac{\pi^2 j^2}{3})^{1/\alpha(u)} \gamma_i \Gamma_i^{-1/\alpha(u)} (|t-V_i|^{H(u)-1/\alpha(u)} - |V_i|^{H(u)-1/\alpha(u)}) \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(V_i) \tag{IV.1}$$

and the linear multistable multifractional motion

$$Y(t) = X(t,t).$$

Then for all $t \in \mathbb{R}$ *and* $\eta < c$*, when* ε *tends to 0,*

$$\mathsf{E}\left[|Y(t+\varepsilon) - Y(t)|^\eta\right] \sim \frac{2^{\eta-1}\Gamma(1-\frac{\eta}{\alpha(t)})}{\eta\int_0^\infty u^{-\eta-1}\sin^2(u)du} \left(\int_{\mathbb{R}} \left||1-x|^{H(t)-\frac{1}{\alpha(t)}} - |x|^{H(t)-\frac{1}{\alpha(t)}}\right|^{\alpha(t)} dx\right)^{\frac{\eta}{\alpha(t)}} \varepsilon^{\eta H(t)}.$$

Proof
See Section IV.3.

Theorem IV.4. *Let* Y *be the linear multistable multifractional motion defined on* $\mathbb{R}$ *with* $H - \frac{1}{\alpha}$ *a non-negative function. For all* $t \in \mathbb{R}$*, almost surely,*

$$\mathcal{H}_t \leq H(t).$$

Proof
See Section IV.3.

IV.1.4 Example : the Lévy multistable motion

In the case of the Lévy multistable motion, we are able to provide a more precise result, to the effect that, at each point, the exact almost sure value of the Hölder exponent is known. Let us first recall some definitions. With M again denoting a symmetric α-stable ($0 < \alpha < 2$) random measure on $\mathbb{R}$ with control measure Lebesgue measure $\mathcal{L}$, we write

$$L_\alpha(t) := \int_0^t M(dz)$$

for α-stable Lévy motion.

The localisability of Lévy motion is a consequence of the fact that it is $1/\alpha$-self-similar with stationary increments [14]. Its multistable version and incremental moments are described in the following theorem :

Theorem IV.5 (Symmetric multistable Lévy motion)**.** *Let* $\alpha : [0,1] \to [c,d] \subset (1,2)$ *be continuously differentiable. Let* $(\Gamma_i)_{i\geq 1}$ *be a sequence of arrival times of a Poisson process with unit arrival time,* $(V_i)_{i\geq 1}$ *be a sequence of i.i.d. random variables with distribution* $\hat{m}(dx) = dx$ *on* $[0,1]$*, and* $(\gamma_i)_{i\geq 1}$ *be a sequence of i.i.d. random variables with distribution* $P(\gamma_i = 1) =$

$P(\gamma_i = -1) = 1/2$. Assume finally that the three sequences $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, and $(\gamma_i)_{i\geq 1}$ are independent and define

$$X(t,u) = C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} \mathbf{1}_{[0,t]}(V_i) \tag{IV.2}$$

and the symmetric multistable Lévy motion

$$Y(t) = X(t,t).$$

Then for all $t \in (0,1)$ and $\eta < c$, when ε tends to 0,

$$\mathsf{E}\left[|Y(t+\varepsilon) - Y(t)|^{\eta}\right] \sim \frac{2^{\eta-1}\Gamma(1-\frac{\eta}{\alpha(t)})}{\eta \int_0^{\infty} u^{-\eta-1}\sin^2(u)du}\, \varepsilon^{\frac{\eta}{\alpha(t)}}.$$

Proof
See Section IV.3.

Theorem IV.6. *Let Y be the symmetric multistable Lévy motion defined on $(0,1)$ with $\alpha : [0,1] \to [c,d] \subset (0,2)$. For all $t \in (0,1)$, almost surely,*

$$\mathcal{H}_t \leq \frac{1}{\alpha(t)}.$$

Proof
See Section IV.3.

Theorem IV.7. *Let $u \in U \subset (0,1)$.*

1. *If $0 < \alpha(u) < 1$, almost surely,*

$$\mathcal{H}_u = \min\left(\frac{1}{\alpha(u)}, \mathcal{H}_u^{\alpha}\right),$$

   where $\mathcal{H}_u^{\alpha}$ denotes the Hölder exponent of α at u, at least when $\frac{1}{\alpha(u)} \neq \mathcal{H}_u^{\alpha}$.

2. *If $1 \leq \alpha(u) < 2$, and α is $\mathcal{C}^1$, almost surely,*

$$\mathcal{H}_u = \frac{1}{\alpha(u)}.$$

Proof
See Section IV.4.

Thus, in the case $0 < \alpha(u) < 1$, the regularity of the multistable Lévy motion is the smallest number between $\frac{1}{\alpha(u)}$ and the regularity of the function α at u. This is very similar to the case of the multifractional Brownian motion, where the Hölder exponent is the minimum between the functional parameter h and its regularity [22, 23]. We conjecture that the same result holds even when $\alpha \geq 1$.

IV.2 Intermediate results

Let φ_X denote the characteristic function of the random variable X. We first state the following almost obvious fact :

Proposition IV.8. Assume that for a given $t \in \mathbb{R}$ there exists $\varepsilon_0 > 0$ such that

$$\sup_{r \in B(0,\varepsilon_0)} \int_0^{+\infty} \left| \varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) \right| dv < +\infty,$$

where Y is a symmetrical process. Then there exists $K > 0$ which depends only on t and ε_0 such that for all $x > 0$, and all $r \in (0, \varepsilon_0)$,

$$\mathsf{P}\left(|Y(t+r) - Y(t)| < x\right) \leq K \frac{x}{r^{h(t)}}.$$

If furthermore we suppose that $\sup_{t \in U} \sup_{r \in B(0,\varepsilon_0)} \int_0^{+\infty} \left| \varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) \right| dv < +\infty$, then for all $t \in U$, for all $r \in (0, \varepsilon_0)$, $\mathsf{P}\left(|Y(t+r) - Y(t)| < x\right) \leq K \frac{x}{r^{h(t)}}$.

Proof

This is a straightforward consequence of the inversion formula. Let $x > 0$ and $r < \varepsilon_0$. Since Y is a symmetrical process, $\varphi_{Y(t+r)-Y(t)}$ is an even function and

$$\begin{aligned}
\mathsf{P}\left(|Y(t+r) - Y(t)| < x\right) &= \frac{1}{\pi} \left| \int_0^{+\infty} \varphi_{Y(t+r)-Y(t)}\left(\frac{v}{r^{h(t)}}\right) \sin\left(\frac{vx}{r^{h(t)}}\right) \frac{dv}{v} \right| \\
&\leq \frac{1}{\pi} \frac{x}{r^{h(t)}} \sup_{r \in B(0,\varepsilon_0)} \int_0^{+\infty} \left| \varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) \right| dv \\
&\leq K \frac{x}{r^{h(t)}} \quad \blacksquare
\end{aligned}$$

We now consider multistable processes, first in the finite measure space case, and then in the σ-finite measure space case :

Proposition IV.9. Assuming (C1), (C2) and (C3), there exists $K_U > 0$ such that for all $u \in U$, $v \in U$ and $x > 0$,

$$\mathsf{P}\left(|X(v,v) - X(v,u)| > x\right) \leq K_U \left(\frac{|v-u|^d}{x^d} (1 + |\log \frac{|v-u|}{x}|^d) + \frac{|v-u|^c}{x^c} (1 + |\log \frac{|v-u|}{x}|^c) \right).$$

Proof

See Section IV.3.

In the σ-finite space case, a similar property holds :

Proposition IV.10. Assuming (C1), (Cs2), (Cs3) and (Cs4), there exists $K_U > 0$ such that for all $u \in U$, $v \in U$ and $x > 0$,

$$\mathsf{P}\left(|X(v,v) - X(v,u)| > x\right) \leq K_U \left(\frac{|v-u|^d}{x^d} (1 + |\log \frac{|v-u|}{x}|^d) + \frac{|v-u|^c}{x^c} (1 + |\log \frac{|v-u|}{x}|^c) \right).$$

Proof
We shall apply Proposition IV.9 to the function $g(t,w,x) = r(x)^{1/\alpha(w)} f(t,w,x)$ on $(E,\mathcal{E},\hat{m})$.

- By (C1), the family of functions $v \to f(t,v,x)$ is differentiable for all (v,t) in U^2 and almost all x in E thus $v \to g(t,v,x)$ is differentiable too i.e (C1) holds for g.
- Choose $\delta > \frac{d}{c} - 1$ such that (Cs2) holds.

$$\sup_{w\in U}(|g(t,w,x)|^{\alpha(w)}) = r(x) \sup_{w\in U}(|f(t,w,x)|^{\alpha(w)}).$$

  One has

$$\begin{aligned}\int_{\mathbb{R}} \left[\sup_{w\in U}(|g(t,w,x)|^{\alpha(w)})\right]^{1+\delta} \hat{m}(dx) &= \int_{\mathbb{R}} r(x)^{1+\delta} \left[\sup_{w\in U}(|f(t,w,x)|^{\alpha(w)})\right]^{1+\delta} \hat{m}(dx) \\ &= \int_{\mathbb{R}} \left[\sup_{w\in U}(|f(t,w,x)|^{\alpha(w)})\right]^{1+\delta} r(x)^{\delta} m(dx)\end{aligned}$$

  thus (C2) holds.
- Choose $\delta > \frac{d}{c} - 1$ such that (Cs3) and (Cs4) hold.

$$g'_u(t,w,x) = r(x)^{1/\alpha(w)}(f'_u(t,w,x) - \frac{\alpha'(w)}{\alpha^2(w)} \log(r(x)) f(t,w,x))$$

  and

$$\int_{\mathbb{R}} \left[\sup_{w\in U}(|g'_u(t,w,x)|^{\alpha(w)})\right]^{1+\delta} \hat{m}(dx)$$

$$\leq \int_{\mathbb{R}} \left[\sup_{w\in U}\left[|f'_u(t,w,x) - \frac{\alpha'(w)}{\alpha^2(w)} \log(r(x)) f(t,w,x)|^{\alpha(w)}\right]\right]^{1+\delta} r(x)^{\delta} m(dx).$$

  The inequality $|a+b|^{\delta} \leq \max(1, 2^{\delta-1})(|a|^{\delta} + |b|^{\delta})$ shows that (C3) holds.
  Proposition IV.9 allows to conclude ■

Proposition IV.11. We suppose that there exists a function h defined on U such that (C8), (C10) and (C14) hold. Assuming (C1), (C6), (C7), (C11), (C12), (C13), (C15), one has :

$$\sup_{r\in B(0,\varepsilon)} \int_0^{+\infty} \varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v)dv < +\infty.$$

If in addition we suppose (Cu8), (Cu10), (Cu11), (Cu12), (Cu14) and (Cu15), then

$$\sup_{t\in U} \sup_{r\in B(0,\varepsilon)} \int_0^{+\infty} \varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v)dv < +\infty.$$

Proof
The expression of the characteristic function $\varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}$ is given in [29] :

$$\varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) = \exp\left(-2\int_{\mathbb{R}}\int_0^{+\infty}\sin^2\left(\frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)}f(t+r,t+r,x)}{2r^{h(t)}y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)}f(t,t,x)}{2r^{h(t)}y^{1/\alpha(t)}}\right)dy\, m(dx)\right).$$

For $v \le 1$, $\varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) \le 1$. For $v \ge 1$, we fix $\varepsilon < \frac{1}{d}$. Lemma (IV.14) entails that there exists $K_U > 0$ such that

$$\varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) \le \exp\left(-\int_{\mathbb{R}}\int_{\frac{K_U v^{\frac{d}{1-\varepsilon d}}}{r}}\left|\frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)}f(t+r,t+r,x)}{2r^{h(t)}y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)}f(t,t,x)}{2r^{h(t)}y^{1/\alpha(t)}}\right|^2 dy\, m(dx)\right).$$

Let

$$N(v,t,r) = \int_{\mathbb{R}}\int_{\frac{K_U v^{\frac{d}{1-\varepsilon d}}}{r}}\left|\frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)}f(t+r,t+r,x)}{2r^{h(t)}y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)}f(t,t,x)}{2r^{h(t)}y^{1/\alpha(t)}}\right|^2 dy\, m(dx).$$

Using Lemma (*IV*.15), there exist $K_U > 0$ and $\varepsilon_0 > 0$ such that for all $v \ge 1$,

$$N(v,t,r) \ge K_U v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{c})}.$$

The inequality becomes

$$\varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) \le \exp\left(-K_U v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{c})}\right),$$

and

$$\begin{aligned}\int_0^{+\infty}\varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v)dv &\le 1+\int_1^{\infty}\exp\left(-K_U v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{c})}\right)dv \\ &< +\infty \quad \blacksquare\end{aligned}$$

IV.3 Proofs and technical results

Proof of Proposition IV.9
We proceed as in [29]. Note that condition **(C2)** implies that there exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t\in U}\int_{\mathbb{R}}\left[\sup_{w\in U}\left[|f(t,w,x)\log|f(t,w,x)||^{\alpha(w)}\right]\right]^{1+\delta}\hat{m}(dx) < \infty. \tag{IV.3}$$

The function $u \mapsto C_{\alpha(u)}^{1/\alpha(u)}$ is a C^1 function since $\alpha(u)$ ranges in $[c,d] \subset (0,2)$. We shall denote $a(u) = (m(E))^{1/\alpha(u)}C_{\alpha(u)}^{1/\alpha(u)}$. The function a is thus also C^1. Let $(u,v) \in U^2$. We estimate :

$$X(v,v) - X(v,u) = \sum_{i=1}^{\infty} \gamma_i(\Phi_i(v) - \Phi_i(u)) + \sum_{i=1}^{\infty} \gamma_i(\Psi_i(v) - \Psi_i(u)),$$

where

$$\Phi_i(w) = a(w) i^{-1/\alpha(w)} f(v,w,V_i)$$

and

$$\Psi_i(w) = a(w)\left(\Gamma_i^{-1/\alpha(w)} - i^{-1/\alpha(w)}\right) f(v,w,V_i).$$

Thanks to the assumptions on a and f, Φ_i and Ψ_i are differentiable and one computes :

$$\Phi_i'(w) = a'(w) i^{-1/\alpha(w)} f(v,w,V_i) + a(w) i^{-1/\alpha(w)} f_w'(v,w,V_i) + a(w)\frac{\alpha'(w)}{\alpha(w)^2}\log(i) i^{-1/\alpha(w)} f(v,w,V_i),$$

and

$$\Psi_i'(w) = a'(w)\left(\Gamma_i^{-1/\alpha(w)} - i^{-1/\alpha(w)}\right) f(v,w,V_i) + a(w)\left(\Gamma_i^{-1/\alpha(w)} - i^{-1/\alpha(w)}\right) f_w'(v,w,V_i)$$

$$+a(w)\frac{\alpha'(w)}{\alpha(w)^2}\left(\log(\Gamma_i)\Gamma_i^{-1/\alpha(w)} - \log(i) i^{-1/\alpha(w)}\right) f(v,w,V_i).$$

Consider now the function $h_i : x \to \Phi_i(x) - \Phi_i(u) - \dfrac{\Phi_i(v) - \Phi_i(u)}{v-u}(x-u)$, and the set $K_i = \{x \in [u,v] : h_i'(x) = 0\}$.

The mean value theorem yields that K_i is a non-empty closed set of $\mathbb{R}$. We define then

$$w_i = \min K_i.$$

Considering the function $k_i : x \to \Psi_i(x) - \Psi_i(u) - \dfrac{\Psi_i(v) - \Psi_i(u)}{v-u}(x-u)$, and the set $F_i = \{x \in [u,v] : k_i'(x) = 0\}$, we define also

$$x_i = \min F_i.$$

Then there exists a sequence of independent measurable random numbers $w_i \in [u,v]$ (or $[v,u]$) and a sequence of measurable random numbers $x_i \in [u,v]$ (or $[v,u]$) such that :

$$X(v,u) - X(v,v) = (u-v)\sum_{i=1}^{\infty}(Z_i^1 + Z_i^2 + Z_i^3) + (u-v)\sum_{i=1}^{\infty}(Y_i^1 + Y_i^2 + Y_i^3), \tag{IV.4}$$

where

$$Z_i^1 = \gamma_i a'(w_i) i^{-1/\alpha(w_i)} f(v,w_i,V_i),$$

$$Z_i^2 = \gamma_i a(w_i) i^{-1/\alpha(w_i)} f_u'(v,w_i,V_i),$$

$$Z_i^3 = \gamma_i a(w_i)\frac{\alpha'(w_i)}{\alpha(w_i)^2}\log(i) i^{-1/\alpha(w_i)} f(v,w_i,V_i),$$

$$Y_i^1 = \gamma_i a'(x_i)\left(\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}\right) f(v, x_i, V_i),$$
$$Y_i^2 = \gamma_i a(x_i)\left(\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}\right) f'_u(v, x_i, V_i),$$
$$Y_i^3 = \gamma_i a(x_i)\frac{\alpha'(x_i)}{\alpha(x_i)^2}\left(\log(\Gamma_i)\Gamma_i^{-1/\alpha(x_i)} - \log(i) i^{-1/\alpha(x_i)}\right) f(v, x_i, V_i).$$

Note that each w_i depends on a, f, α, u, v, V_i, and each x_i depends on $a, f, \alpha, u, v, V_i, \Gamma_i$ but not on γ_i. This remark will be useful in the sequel.

In [29], it is proved that each series $\sum_{i=1}^{\infty} Z_i^j$ and $\sum_{i=1}^{\infty} Y_i^j$, $j = 1, 2, 3$, converges almost surely. Let $x > 0$. We consider $\mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j| > x\right)$ and $\mathsf{P}\left(|\sum_{i=1}^{\infty} Y_i^j| > x\right)$ for $j = 1, 2, 3$.

Let $\eta \in (0, \min(\frac{2c}{d} - 1, \frac{c}{d}(\delta + 1) - 1))$. Markov inequality yields

$$\begin{aligned}
\mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j| > x\right) &\leq \frac{1}{x^d}\mathsf{E}\left[|\sum_{i=1}^{\infty} Z_i^j|^d\right] \\
&\leq \frac{1}{x^d}\left(\mathsf{E}\left[|\sum_{i=1}^{\infty} Z_i^j|^{d(1+\eta)}\right]\right)^{\frac{1}{1+\eta}}.
\end{aligned}$$

The random variables Z_i^j are independent with mean 0 thus, by Theorem 2 of [57] :

$$\mathsf{E}\left[|\sum_{i=1}^{+\infty} Z_i^j|^{d(1+\eta)}\right] \leq 2\sum_{i=1}^{+\infty} \mathsf{E}[|Z_i^j|^{d(1+\eta)}].$$

For $j = 1$,

$$\begin{aligned}
\mathsf{E}\left[|Z_i^1|^{d(1+\eta)}\right] &= \mathsf{E}\left[|a'(w_i)|^{d(1+\eta)} i^{-\frac{d(1+\eta)}{\alpha(w_i)}} |f(v, w_i, V_i)|^{d(1+\eta)}\right] \\
&\leq \frac{K_U}{i^{1+\eta}}\mathsf{E}\left[\left(\sup_{w\in B(u,\varepsilon)} |f(v, w, V_i)|^{\alpha(w)}\right)^{\frac{d(1+\eta)}{\alpha(w_i)}}\right] \\
&\leq \frac{K_U}{i^{1+\eta}}\mathsf{E}\left[(\sup_{w\in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)})^{1+\eta} + (\sup_{w\in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)})^{\frac{d}{c}(1+\eta)}\right] \\
&\leq \frac{K_U}{i^{1+\eta}}.
\end{aligned}$$

For $j = 2$,

$$\begin{aligned}
\mathsf{E}\left[|Z_i^2|^{d(1+\eta)}\right] &\leq \frac{K_U}{i^{1+\eta}}\mathsf{E}\left[(\sup_{w\in B(u,\varepsilon)} |f'_u(v, w, V_1)|^{\alpha(w)})^{1+\eta} + (\sup_{w\in B(u,\varepsilon)} |f'_u(v, w, V_1)|^{\alpha(w)})^{\frac{d}{c}(1+\eta)}\right] \\
&\leq \frac{K_U}{i^{1+\eta}}.
\end{aligned}$$

For $j=3$,

$$\begin{aligned}
\mathsf{E}\left[|Z_i^3|^{d(1+\eta)}\right] &= \mathsf{E}\left[|a(w_i)\frac{\alpha'(w_i)}{\alpha(w_i)^2}|^{d(1+\eta)}|f(v,w_i,V_i)|^{d(1+\eta)}\frac{(\log i)^{d(1+\eta)}}{i^{\frac{d(1+\eta)}{\alpha(w_i)}}}\right] \\
&\leq K_U\frac{(\log i)^{d(1+\eta)}}{i^{1+\eta}}.
\end{aligned}$$

Finally, $\sup_{v\in U}\sum_{i=1}^{+\infty}\mathsf{E}\left[|Z_i^j|^{d(1+\eta)}\right] < +\infty$ thus

$$\mathsf{P}\left(|\sum_{i=1}^{\infty}Z_i^j| > x\right) \leq \frac{K_U}{x^d}.$$

We consider now $\mathsf{P}\left(|\sum_{i=1}^{\infty}Y_i^j| > x\right)$ for $j=1,2,3$.

$$\mathsf{P}\left(|\sum_{i=1}^{\infty}Y_i^j| > x\right) \leq \mathsf{P}\left(|Y_1^j| \geq \frac{x}{2}\right) + \mathsf{P}\left(|\sum_{i=2}^{\infty}Y_i^j| \geq \frac{x}{2}\right).$$

Since $\mathsf{P}\left(|\sum_{i=2}^{\infty}Y_i^j| \geq \frac{x}{2}\right) \leq \frac{2^d}{x^d}\left(\mathsf{E}\left[|\sum_{i=2}^{\infty}Y_i^j|^{d(1+\eta)}\right]\right)^{\frac{1}{1+\eta}}$, we want to apply Theorem 2 of [57] again. Let $S_m = \sum_{i=1}^{m}Y_i^j$ and write $Y_i^j = \gamma_i W_i^j$. Note that γ_i is independent of W_i^j and S_{i-1}.

$$\begin{aligned}
\mathsf{E}\left(Y_{m+1}^j|S_m\right) &= \mathsf{E}\left(\mathsf{E}(Y_{m+1}^j|S_m,W_{m+1})|S_m\right) \\
&= \mathsf{E}\left(\mathsf{E}(\gamma_{m+1}W_{m+1}^j|S_m,W_{m+1})|S_m\right) \\
&= \mathsf{E}\left(W_{m+1}^j\mathsf{E}(\gamma_{m+1}|S_m,W_{m+1})|S_m\right) \\
&= \mathsf{E}\left(W_{m+1}^j\mathsf{E}(\gamma_{m+1})|S_m\right) \\
&= 0.
\end{aligned}$$

We apply Theorem 2 of [57] with $(d(1+\eta)<2)$,

$$\mathsf{E}\left[|\sum_{i=2}^{\infty}Y_i^j|^{d(1+\eta)}\right] \leq 2\sum_{i=2}^{\infty}\mathsf{E}|Y_i^j|^{d(1+\eta)},$$

and

$$\mathsf{P}\left(|\sum_{i=1}^{\infty}Y_i^j| > x\right) \leq \mathsf{P}\left(|Y_1^j| \geq \frac{x}{2}\right) + \frac{2^d}{x^d}\left(2\sum_{i=2}^{\infty}\mathsf{E}|Y_i^j|^{d(1+\eta)}\right)^{\frac{1}{1+\eta}}.$$

For $j=1$,

$$
\begin{aligned}
\mathsf{P}\left(|Y_1^1| \geq \frac{x}{2}\right) &= \mathsf{P}\left(|a'(x_1)|^{\alpha(x_1)}|\frac{1}{\Gamma_1^{1/\alpha(x_1)}} - 1|^{\alpha(x_1)}|f(v,x_1,V_1)|^{\alpha(x_1)} \geq \frac{x^{\alpha(x_1)}}{2^{\alpha(x_1)}}\right) \\
&\leq \mathsf{P}\left(|\frac{1}{\Gamma_1^{1/\alpha(x_1)}} - 1|^{\alpha(x_1)} \sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)} \geq K_U x^{\alpha(x_1)}\right).
\end{aligned}
$$

For $x < 1$,

$$
\begin{aligned}
\mathsf{P}\left(|Y_1^1| \geq \frac{x}{2}\right) &\leq \mathsf{P}\left(|\frac{1}{\Gamma_1^{1/\alpha(x_1)}} - 1|^{\alpha(x_1)} \sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)} \geq K_U x^d\right) \\
&\leq \mathsf{P}\left(\{\sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)} \geq K_U x^d\} \cap \{\Gamma_1 > 1\}\right) \\
&+ \mathsf{P}\left(\{|\frac{1}{\Gamma_1}|\Gamma_1^{1/\alpha(x_1)} - 1|^{\alpha(x_1)} \sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)} \geq K_U x^d\} \cap \{\Gamma_1 \leq 1\}\right).
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{P}\left(\{\sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)} \geq K_U x^d\} \cap \{\Gamma_1 > 1\}\right) &\leq \frac{K_U}{x^d}\mathsf{E}\left(\sup_{w\in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)}\right) \\
&\leq \frac{K_U}{x^d}.
\end{aligned}
$$

Let $W(v,x) = \sup_{w\in B(u,\varepsilon)} |f(v,w,x)|^{\alpha(w)}$ and F_{v,V_1} be the distribution of $W(v,V_1)$.

$$
\begin{aligned}
\mathsf{P}\left(\{|\frac{1}{\Gamma_1}|\Gamma_1^{1/\alpha(x_1)} - 1|^{\alpha(x_1)} W(v,V_1) \geq K_U x^d\} \cap \{\Gamma_1 \leq 1\}\right) &\leq \mathsf{P}\left(W(v,V_1) \geq K_U x^d \Gamma_1\right) \\
&= \int_0^{+\infty} \mathsf{P}\left(z \geq K_U x^d \Gamma_1\right) F_{v,V_1}(dz) \\
&= \int_0^{+\infty} \left(1 - e^{-\frac{z}{K_U x^d}}\right) F_{v,V_1}(dz) \\
&\leq \int_0^{+\infty} \frac{z}{K_U x^d} F_{v,V_1}(dz) \\
&\leq \frac{K_U}{x^d}.
\end{aligned}
$$

For $x \geq 1$,

$$\begin{aligned}
\mathsf{P}\left(|Y_1^1| \geq \frac{x}{2}\right) &\leq \mathsf{P}\left(|\frac{1}{\Gamma_1^{1/\alpha(x_1)}} - 1|^{\alpha(x_1)} \sup_{w \in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)} \geq K_U x^c\right) \\
\mathsf{P}\left(|Y_1^1| \geq \frac{x}{2}\right) &\leq \frac{K_U}{x^c}.
\end{aligned}$$

For $i \geq 2$,

$$\begin{aligned}
\mathsf{E}|Y_i^1|^{d(1+\eta)} &= \mathsf{E}\left(|a'(x_i)|^{d(1+\eta)} |\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}|^{d(1+\eta)} \left(|f(v,x_i,V_i)|^{\alpha(x_i)}\right)^{\frac{d(1+\eta)}{\alpha(x_i)}}\right) \\
&\leq K_U \mathsf{E}\left(i^{-\frac{d(1+\eta)}{\alpha(x_i)}} W(v,V_i)^{\frac{d(1+\eta)}{\alpha(x_i)}} |(\frac{i}{\Gamma_i})^{1/\alpha(x_i)} - 1|^{d(1+\eta)}\right) \\
&\leq \frac{K_U}{i^{1+\eta}} \mathsf{E}\left(\left[W(v,V_i)^{1+\eta} + W(v,V_i)^{\frac{d}{c}(1+\eta)}\right]\left[|(\frac{i}{\Gamma_i})^{1/c} - 1|^{d(1+\eta)} + |(\frac{i}{\Gamma_i})^{1/d} - 1|^{d(1+\eta)}\right]\right) \\
&\leq \frac{K_U}{i^{1+\eta}} \mathsf{E}\left(W(v,V_i)^{1+\eta} + W(v,V_i)^{\frac{d}{c}(1+\eta)}\right) \mathsf{E}\left(|(\frac{i}{\Gamma_i})^{1/c} - 1|^{d(1+\eta)} + |(\frac{i}{\Gamma_i})^{1/d} - 1|^{d(1+\eta)}\right).
\end{aligned}$$

Using the fact that $\eta \leq \delta$ and $\frac{d}{c}(1+\eta) \leq 1+\delta$,

$$\begin{aligned}
\mathsf{E}\left(W(v,V_i)^{1+\eta} + W(v,V_i)^{\frac{d}{c}(1+\eta)}\right) &= \mathsf{E}\left(W(v,V_1)^{1+\eta} + W(v,V_1)^{\frac{d}{c}(1+\eta)}\right) \\
&\leq K_U,
\end{aligned}$$

$$\begin{aligned}
\mathsf{E}|(\frac{i}{\Gamma_i})^{1/c} - 1|^{d(1+\eta)} &\leq K_U(1 + \mathsf{E}\left((\frac{i}{\Gamma_i})^{\frac{d}{c}(1+\eta)}\right)) \\
&\leq K_U,
\end{aligned}$$

and

$$\mathsf{E}|(\frac{i}{\Gamma_i})^{1/d} - 1|^{d(1+\eta)} \leq K_U.$$

As a consequence :

$$\sup_{v \in U} \sum_{i=2}^{+\infty} \mathsf{E}|Y_i^1|^{d(1+\eta)} \leq K_U$$

and

$$\mathsf{P}\left(\left|\sum_{i=1}^{\infty} Y_i^1\right| > x\right) \leq K_U \left(\frac{1}{x^c} + \frac{1}{x^d}\right).$$

For $j = 2$, since the conditions required on (a', f) are also satisfied by (a, f'_u), one gets in a similar fashion

$$\mathsf{P}\left(\left|\sum_{i=1}^{\infty} Y_i^2\right| > x\right) \leq K_U \left(\frac{1}{x^c} + \frac{1}{x^d}\right).$$

For $j = 3$,

$$\begin{aligned}\mathsf{P}\left(|Y_1^3| \geq \frac{x}{2}\right) &= \mathsf{P}\left(|a(x_1)\frac{\alpha'(x_1)}{\alpha(x_1)^2}\log(\Gamma_1)\Gamma_1^{-1/\alpha(x_1)}f(v,x_1,V_1)| \geq \frac{x}{2}\right)\\ &\leq \mathsf{P}\left(K_U\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \geq \frac{\Gamma_1}{|\log\Gamma_1|^{\alpha(x_1)}}\right).\end{aligned}$$

Let $g(z) = \frac{z}{|\log z|^{\alpha(x_1)}}$, for $z < 1$.

g is a one-to-one increasing function, and for all $z < 1$ such that $z|\log z|^{\alpha(x_1)} < 1$ and $|1+\alpha(x_1)\frac{\log|\log z|}{|\log z|}|^{\alpha(x_1)} \leq 2$,

$$g\left(z|\log z|^{\alpha(x_1)}\right) = \frac{z|\log z|^{\alpha(x_1)}}{|\log z+\alpha(x_1)\log|\log z||^{\alpha(x_1)}} \geq \frac{z}{2}$$

thus $g^{-1}(\frac{z}{2}) \leq z|\log z|^{\alpha(x_1)}$.
Fix $A > 0$ such that for all $0 < z < A$, $g^{-1}(z) \leq 2z|\log 2+\log z|^{\alpha(x_1)}$ *i.e.*

$$g^{-1}(z) \leq K_U z|\log z|^{\alpha(x_1)}.$$

Let $B = \{K_U\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \geq \frac{\Gamma_1}{|\log\Gamma_1|^{\alpha(x_1)}}\}$.

$$\mathsf{P}(B) = \mathsf{P}(B\cap\{\Gamma_1>1\}) + \mathsf{P}(B\cap\{\Gamma_1<1\}\cap\{0\leq K_U\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \leq A\})$$

$$+\mathsf{P}(B\cap\{\Gamma_1<1\}\cap\{K_U\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} > A\}).$$

Each of these three terms will be treated separately.

- $$\begin{aligned}\mathsf{P}\left(B\cap\{\Gamma_1>1\}\right) &\leq \mathsf{P}\left(K_U\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}}|\log\Gamma_1|^{\alpha(x_1)} \geq 1\right)\\ &\leq \mathsf{P}\left(K_U\sup_{w\in B(u,\varepsilon)}|f(v,w,V_1)|^{\alpha(w)}(|\log\Gamma_1|^c+|\log\Gamma_1|^d) \geq x^{\alpha(x_1)}\right).\end{aligned}$$

For $x \geq 1$,

$$\begin{aligned}\mathsf{P}\left(B\cap\{\Gamma_1>1\}\right) &\leq \mathsf{P}\left(K_U\sup_{w\in B(u,\varepsilon)}|f(v,w,V_1)|^{\alpha(w)}(|\log\Gamma_1|^c+|\log\Gamma_1|^d) \geq x^c\right)\\ &\leq \frac{K_U}{x^c}\mathsf{E}\left(\sup_{w\in B(u,\varepsilon)}|f(v,w,V_1)|^{\alpha(w)}\right)\mathsf{E}\left(|\log\Gamma_1|^c+|\log\Gamma_1|^d\right)\\ &\leq \frac{K_U}{x^c}.\end{aligned}$$

For $x<1$,

$$\mathsf{P}\left(B\cap\{\Gamma_1>1\}\right) \;\leq\; \mathsf{P}\left(K_U \sup_{w\in B(u,\varepsilon)}|f(v,w,V_1)|^{\alpha(w)}(|\log\Gamma_1|^c+|\log\Gamma_1|^d)\geq x^d\right) \;\leq\; \frac{K_U}{x^d}.$$

- $$\mathsf{P}\left(B\cap\{\Gamma_1<1\}\cap\{K_U\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}}>A\}\right) \;\leq\; \mathsf{P}\left(K_U|f(v,x_1,V_1)|^{\alpha(x_1)}\geq Ax^{\alpha(x_1)}\right) \;\leq\; \frac{K_U}{x^c}+\frac{K_U}{x^d}.$$

- $$\mathsf{P}\left(B\cap\{\Gamma_1<1\}\cap\{0\leq K_U\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}}\leq A\}\right)$$

$$\begin{aligned}
&= \mathsf{P}\left(\{g(\Gamma_1)\leq K_U\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}}\}\cap\{\Gamma_1<1\}\cap\{0\leq K_U\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}}\leq A\}\right)\\
&\leq \mathsf{P}\left(\Gamma_1\leq K_U\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}}+K_U\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}}\left|\log\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}}\right|^{\alpha(x_1)}\right)\\
&\leq \mathsf{P}\left(\Gamma_1\leq K_U|f(v,x_1,V_1)|^{\alpha(x_1)}(\frac{1+|\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}})+K_U\frac{||f(v,x_1,V_1)|\log|f(v,x_1,V_1)||^{\alpha(x_1)}}{x^{\alpha(x_1)}}\right).
\end{aligned}$$

With $W(v,x)=\sup_{w\in B(u,\varepsilon)}|f(v,w,x)|^{\alpha(w)}$ and $Z(v,x)=\sup_{w\in B(u,\varepsilon)}|f(v,w,x)\log|f(v,w,x)||^{\alpha(w)}$,

$$\mathsf{P}\left(B\cap\{\Gamma_1<1\}\cap\{0\leq K_U\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}}\leq A\}\right)$$

$$\begin{aligned}
&\leq \mathsf{P}\left(\Gamma_1\leq K_UW(v,V_1)(\frac{1+|\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}})+K_U\frac{Z(v,V_1)}{x^{\alpha(x_1)}}\right)\\
&\leq \mathsf{P}\left(\Gamma_1\leq K_UW(v,V_1)(\frac{1+|\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}})\right)+\mathsf{P}\left(\Gamma_1\leq K_UZ(v,V_1)(\frac{1+|\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}})\right).
\end{aligned}$$

Since $\frac{1+|\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \leq K_U(\frac{1+|\log x|^c}{x^c} + \frac{1+|\log x|^d}{x^d})$,

$$\begin{aligned}
\mathsf{P}\left(\Gamma_1 \leq K_U W(v,V_1)(\frac{1+|\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}})\right) &\leq \mathsf{P}\left(\Gamma_1 \leq K_U W(v,V_1)(\frac{1+|\log x|^c}{x^c} + \frac{1+|\log x|^d}{x^d})\right) \\
&\leq K_U(\frac{1+|\log x|^c}{x^c} + \frac{1+|\log x|^d}{x^d}),
\end{aligned}$$

and

$$\mathsf{P}\left(\Gamma_1 \leq K_U Z(v,V_1)(\frac{1+|\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}})\right) \leq \mathsf{P}\left(\Gamma_1 \leq K_U Z(v,V_1)(\frac{1+|\log x|^c}{x^c} + \frac{1+|\log x|^d}{x^d})\right).$$

Denoting G_{v,V_1} the distribution of $Z(v,V_1)$,

$$\begin{aligned}
&\mathsf{P}\left(\Gamma_1 \leq K_U Z(v,V_1)(\frac{1+|\log x|^c}{x^c} + \frac{1+|\log x|^d}{x^d})\right) \\
&= \int_0^{+\infty} (1-\exp(-K_U(\frac{1+|\log x|^c}{x^c} + \frac{1+|\log x|^d}{x^d})z))G_{v,V_1}(dz) \\
&\leq K_U(\frac{1+|\log x|^c}{x^c} + \frac{1+|\log x|^d}{x^d}) \int_0^{+\infty} zG_{v,V_1}(dz) \\
&\leq K_U(\frac{1+|\log x|^c}{x^c} + \frac{1+|\log x|^d}{x^d}),
\end{aligned}$$

since $\sup_{v\in B(u,\varepsilon)} \mathsf{E}\left(Z(v,V_1)\right) < +\infty$.

Finally,

$$\mathsf{P}\left(B \cap \{\Gamma_1 < 1\} \cap \{0 \leq K_U \frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \leq A\}\right) \leq K_U(\frac{1+|\log x|^c}{x^c} + \frac{1+|\log x|^d}{x^d})$$

and

$$\mathsf{P}\left(|Y_1^3| \geq \frac{x}{2}\right) \leq K_U(\frac{1+|\log x|^c}{x^c} + \frac{1+|\log x|^d}{x^d}).$$

For $i \geq 2$,

$$\begin{aligned}
\mathsf{E}|Y_i^3|^{d(1+\eta)} &\leq K_U \frac{|\log i|^{d(1+\eta)}}{i^{1+\eta}} \mathsf{E}\left(W(v,V_i)^{1+\eta} + W(v,V_i)^{\frac{d}{c}(1+\eta)} \right) \mathsf{E}\left(|\frac{\log \Gamma_i}{\log i}||(\frac{i}{\Gamma_i})^{1/\alpha(x_i)} - 1|^{d(1+\eta)} \right) \\
&\leq K_U \frac{|\log i|^{d(1+\eta)}}{i^{1+\eta}} \mathsf{E}\left(|\frac{\log \Gamma_i}{\log i}||(\frac{i}{\Gamma_i})^{1/\alpha(x_i)} - 1|^{d(1+\eta)} \right) \\
&\leq K_U \frac{|\log i|^{d(1+\eta)}}{i^{1+\eta}} \mathsf{E}\left(|\frac{\log \Gamma_i}{\log i}||(\frac{i}{\Gamma_i})^{1/c} - 1|^{d(1+\eta)} + |\frac{\log \Gamma_i}{\log i}||(\frac{i}{\Gamma_i})^{1/d} - 1|^{d(1+\eta)} \right) \\
&\leq K_U \frac{|\log i|^{d(1+\eta)}}{i^{1+\eta}},
\end{aligned}$$

thus

$$\sup_{v\in U} \sum_{i=2}^{+\infty} \mathsf{E}|Y_i^3|^{d(1+\eta)} \leq K_U$$

and

$$\mathsf{P}\left(\left| \sum_{i=1}^{\infty} Y_i^3 \right| > x \right) \leq K_U \left(\frac{1+|\log x|^c}{x^c} + \frac{1+|\log x|^d}{x^d}\right).$$

Let us go back to $\mathsf{P}\left(|X(v,v) - X(v,u)| > x\right)$.

$$\begin{aligned}
\mathsf{P}\left(|X(v,v) - X(v,u)| > x\right) &= \mathsf{P}\left(|u-v|| \sum_{i=1}^{\infty} (Z_i^1 + Z_i^2 + Z_i^3 + Y_i^1 + Y_i^2 + Y_i^3)| > x \right) \\
&\leq \sum_{j=1}^{3} \left(\mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j| \geq \frac{x}{6|u-v|} \right) + \mathsf{P}\left(|\sum_{i=1}^{\infty} Y_i^j| \geq \frac{x}{6|u-v|} \right) \right) \\
&\leq K_U \left(\frac{|v-u|^d}{x^d}(1 + |\log \frac{|v-u|}{x}|^d) + \frac{|v-u|^c}{x^c}(1 + |\log \frac{|v-u|}{x}|^c) \right)
\end{aligned}$$

and the proof is complete ■

Lemma IV.12. Assume (C11), (C12), (C14), (C15). There exists a function $l \geq 0$ such that

$$\lim_{r\to 0} |\Delta(r,t) - l(t)| = 0,$$

where

$$\Delta(r,t) =: \frac{1}{r^{2h(t)}} \int_{\mathbb{R}} \int_{\frac{K}{r}} \left| \frac{C_{\alpha(t+r)}^{1/\alpha(t+r)}}{y^{1/\alpha(t+r)}} f(t+r, t+r, x) - \frac{C_{\alpha(t)}^{1/\alpha(t)}}{y^{1/\alpha(t)}} f(t,t,x) \right|^2 dy\, m(dx).$$

Assuming in addition (Cu11), (Cu12), (Cu14), (Cu15), the convergence is uniform on U.

Proof

Let $l(t)=\frac{C_{\alpha(t)}^{2/\alpha(t)}K^{1-\frac{2}{\alpha(t)}}}{\frac{2}{\alpha(t)}-1}g(t)$. Note that condition (C14) implies the following :

$$\forall\varepsilon>0,\exists K_U>0,\forall r\leq\varepsilon,\frac{1}{|r|^{1+2(h(t)-\frac{1}{\alpha(t)})}}\int_{\mathbb{R}}|f(t+r,t,x)-f(t,t,x)|^2\,m(dx)\leq K_U. \quad \text{(IV.5)}$$

The uniform condition (Cu14) implies also that :

$$\exists K_U>0,\forall v\in U,\forall u\in U,\frac{1}{|v-u|^{1+2(h(u)-\frac{1}{\alpha(u)})}}\int_{\mathbb{R}}|f(v,u,x)-f(u,u,x)|^2\,m(dx)\leq K_U. \quad \text{(IV.6)}$$

Expanding the square, we can write $\Delta(r,t)-l(t)=\Delta_1(r,t)+\Delta_2(r,t)+\Delta_3(r,t)$ where

$$\Delta_1(r,t)=\frac{1}{r^{2h(t)}}\int_{\mathbb{R}}\int_{\frac{K}{r}}\left|\frac{C_{\alpha(t+r)}^{1/\alpha(t+r)}}{y^{1/\alpha(t+r)}}f(t+r,t+r,x)-\frac{C_{\alpha(t)}^{1/\alpha(t)}}{y^{1/\alpha(t)}}f(t+r,t,x)\right|^2 dy\,m(dx),$$

$$\Delta_2(r,t)=\frac{2C_{\alpha(t)}^{1/\alpha(t)}}{r^{2h(t)}}\int_{\mathbb{R}}\int_{\frac{K}{r}}\frac{1}{y^{1/\alpha(t)}}g_1(r,t,x,y)g_2(r,t,x)dy\,m(dx),$$

and

$$\Delta_3(r,t)=\frac{1}{r^{2h(t)}}\int_{\mathbb{R}}\int_{\frac{K}{r}}\frac{C_{\alpha(t)}^{2/\alpha(t)}}{y^{2/\alpha(t)}}\left(f(t+r,t,x)-f(t,t,x)\right)^2 dy\,m(dx)-l(t),$$

with $g_1(r,t,x,y)=\frac{C_{\alpha(t+r)}^{1/\alpha(t+r)}}{y^{1/\alpha(t+r)}}f(t+r,t+r,x)-\frac{C_{\alpha(t)}^{1/\alpha(t)}}{y^{1/\alpha(t)}}f(t+r,t,x)$ and $g_2(r,t,x)=f(t+r,t,x)-f(t,t,x)$. Since α is continuous, there exists a positive constant K_U (that may change from line to line) such that

$$\begin{aligned}
|\Delta_2(r,t)| &\leq \frac{K_U}{r^{2h(t)}}\int_{\mathbb{R}}\int_{\frac{K}{r}}\left|\frac{g_1(r,t,x,y)g_2(r,t,x)}{y^{1/\alpha(t)}}\right|dy\,m(dx)\\
&\leq \frac{K_U}{r^{2h(t)}}\left(\int_{\mathbb{R}}\int_{\frac{K}{r}}|g_1(r,t,x,y)|^2\,dy\,m(dx)\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\int_{\frac{K}{r}}\left|\frac{g_2(r,t,x)}{y^{1/\alpha(t)}}\right|^2 dy\,m(dx)\right)^{\frac{1}{2}}\\
&\leq \frac{K_U}{r^{2h(t)}}r^{h(t)}\sqrt{\Delta_1(r,t)}\left(\int_{\mathbb{R}}\int_{\frac{K}{r}}\left|\frac{g_2(r,t,x)}{y^{1/\alpha(t)}}\right|^2 dy\,m(dx)\right)^{\frac{1}{2}}\\
&\leq \frac{K_U}{r^{h(t)}}\sqrt{\Delta_1(r,t)}\left(\int_{\mathbb{R}}|g_2(r,t,x)|^2\,m(dx)\right)^{\frac{1}{2}}r^{\frac{1}{\alpha(t)}-\frac{1}{2}}K^{\frac{1}{2}-\frac{1}{\alpha(t)}}\sqrt{\frac{\alpha(t)}{2-\alpha(t)}}\\
&\leq K_U\sqrt{\Delta_1(r,t)}\left(\frac{1}{r^{1+2(h(t)-\frac{1}{\alpha(t)})}}\int_{\mathbb{R}}|f(t+r,t,x)-f(t,t,x)|^2\,m(dx)\right)^{\frac{1}{2}}\\
&\leq K_U\sqrt{\Delta_1(r,t)}\text{ with }\textbf{(IV.5)}.
\end{aligned}$$

Let us show that $\lim_{r\to 0}\sqrt{\Delta_1(r,t)}=0$. The triangle inequality yields $\sqrt{\Delta_1(r,t))}\le \delta_1(r,t)+\delta_2(r,t)+\delta_3(r,t)$ where

$$\delta_1(r,t)=\frac{1}{2r^{h(t)}}\left(\int_{\mathbb{R}}\int_{\frac{K}{r}}\left|C_{\alpha(t+r)}^{1/\alpha(t+r)}-C_{\alpha(t)}^{1/\alpha(t)}\right|^2\frac{|f(t+r,t+r,x)|^2}{y^{2/\alpha(t+r)}}dy\,m(dx)\right)^{\frac{1}{2}},$$

$$\delta_2(r,t)=\frac{1}{2r^{h(t)}}\left(\int_{\mathbb{R}}\int_{\frac{K}{r}}\frac{C_{\alpha(t)}^{2/\alpha(t)}}{y^{2/\alpha(t+r)}}\left|f(t+r,t+r,x)-f(t+r,t,x)\right|^2dy\,m(dx)\right)^{\frac{1}{2}},$$

and

$$\delta_3(r,t)=\frac{1}{2r^{h(t)}}\left(\int_{\mathbb{R}}\int_{\frac{K}{r}}C_{\alpha(t)}^{2/\alpha(t)}|f(t+r,t,x)|^2\left(\frac{1}{y^{1/\alpha(t+r)}}-\frac{1}{y^{1/\alpha(t)}}\right)^2dy\,m(dx)\right)^{\frac{1}{2}}.$$

Now,

$$\delta_1(r,t)\ \le\ K_U\frac{|C_{\alpha(t+r)}^{1/\alpha(t+r)}-C_{\alpha(t)}^{1/\alpha(t)}|}{r^{h(t)}}\left(\frac{1}{\frac{2}{\alpha(t+r)}-1}(\frac{K}{r})^{1-\frac{2}{\alpha(t+r)}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}|f(t+r,t+r,x)|^2\,m(dx)\right)^{\frac{1}{2}}.$$

Since the function $u\mapsto C_{\alpha(u)}^{1/\alpha(u)}$ is a C^1 function,

$$\begin{aligned}\delta_1(r,t)\ &\le\ K_U r^{1-h(t)+\frac{1}{\alpha(t+r)}-\frac{1}{2}}\left(\int_{\mathbb{R}}|f(t+r,t+r,x)|^2\,m(dx)\right)^{\frac{1}{2}}\\ &\le\ K_U r^{1-h(t)+\frac{1}{\alpha(t+r)}-\frac{1}{2}}\text{ with }\mathbf{(C12)}\\ &\le\ K_U r^{\frac{1}{2}+\frac{1}{d}-h_+}.\end{aligned}$$

Since $\frac{1}{2}+\frac{1}{d}-h_+>0$, $\lim_{r\to 0}\delta_1(r,t)\ =0$.

$$\begin{aligned}\delta_2(r,t)\ &\le\ \frac{C_{\alpha(t)}^{1/\alpha(t)}}{2r^{h(t)}}\left(\frac{1}{\frac{2}{\alpha(t+r)}-1}(\frac{K}{r})^{1-\frac{2}{\alpha(t+r)}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}|f(t+r,t+r,x)-f(t+r,t,x)|^2\,m(dx)\right)^{\frac{1}{2}}\\ &\le\ K_U r^{\frac{1}{\alpha(t+r)}-h(t)-\frac{1}{2}}\left(\int_{\mathbb{R}}|f(t+r,t+r,x)-f(t+r,t,x)|^2\,m(dx)\right)^{\frac{1}{2}}\\ &\le\ K_U r^{\frac{1}{2}+\frac{1}{\alpha(t+r)}-h(t)}\text{ with }\mathbf{(C15)}\\ &\le\ K_U r^{\frac{1}{2}+\frac{1}{d}-h_+},\end{aligned}$$

thus $\lim_{r\to 0}\delta_2(r,t)=0$.

$$\delta_3(r,t) \leq \frac{C_{\alpha(t)}^{1/\alpha(t)}}{2r^{h(t)}} \left(\int_{\mathbb{R}} |f(t+r,t,x)|^2\, m(dx)\right)^{\frac{1}{2}} \left(\int_{\frac{K}{r}} \left(\frac{1}{y^{1/\alpha(t+r)}} - \frac{1}{y^{1/\alpha(t)}}\right)^2 dy\right)^{\frac{1}{2}}$$

Since the function $u \mapsto \alpha(u)$ is a C^1 function, $\forall \eta < \frac{1}{d}$,

$$\begin{aligned}
\delta_3(r,t) &\leq \frac{K_U}{r^{h(t)}} \left(\int_{\mathbb{R}} |f(t+r,t,x)|^2\, m(dx)\right)^{\frac{1}{2}} K_U r^{\frac{1}{2}+\frac{1}{d}-\eta} \\
&\leq K_U r^{\frac{1}{2}+\frac{1}{d}-\eta-h_+} \text{ with } \mathbf{(C11)}
\end{aligned}$$

thus $\lim_{r\to 0} \delta_3(r,t) = 0$. Finally, $\lim_{r\to 0} \sqrt{\Delta_1(r,t)} = 0$.

Let us now consider the last term $\Delta_3(r,t)$:

$$\Delta_3(r,t) = \frac{C_{\alpha(t)}^{2/\alpha(t)} K^{1-\frac{2}{\alpha(t)}}}{\frac{2}{\alpha(t)} - 1} \left(\frac{1}{r^{1+2(h(t)-1/\alpha(t)}} \int_{\mathbb{R}} \left(f(t+r,t,x) - f(t,t,x)\right)^2 m(dx) - g(t)\right)$$

thus, with **(C14)**, $\lim_{r\to 0} |\Delta_3(r,t)| = 0$ ■

Lemma IV.13. Assume (C6), (C10), (C12), (C13), (C15), and let :

$$\Delta(r,t) =: \frac{1}{r^{1+2(h(t)-1/\alpha(t))}} \left(\frac{C_{\alpha(t)}^{1/\alpha(t)} K^{\frac{1}{\alpha(t+r)}-\frac{1}{\alpha(t)}} \left(\frac{2}{\alpha(t+r)} - 1\right) \int_{\mathbb{R}} f(t+r,t+r,x) f(t,t,x) m(dx)}{C_{\alpha(t+r)}^{1/\alpha(t+r)} r^{\frac{1}{\alpha(t+r)}-\frac{1}{\alpha(t)}} \left(\frac{1}{\alpha(t+r)} + \frac{1}{\alpha(t)} - 1\right) \int_{\mathbb{R}} f(t+r,t+r,x)^2 m(dx)} - 1\right)^2.$$

Then :

$$\lim_{r\to 0} |\Delta(r,t)| = 0.$$

If in addition we suppose (Cu10), (Cu12), (Cu15), the convergence is uniform on U.

Proof

Since the function $t \mapsto \alpha(t)$ is a C^1 function, there exists $K_U > 0$ such that

$$\left|\frac{C_{\alpha(t)}^{1/\alpha(t)}}{C_{\alpha(t+r)}^{1/\alpha(t+r)}} - 1\right| \leq rK_U, \tag{IV.7}$$

$$\left|K^{\frac{1}{\alpha(t+r)}-\frac{1}{\alpha(t)}} - 1\right| \leq rK_U, \tag{IV.8}$$

and

$$\left|\frac{\frac{2}{\alpha(t+r)} - 1}{\frac{1}{\alpha(t+r)} + \frac{1}{\alpha(t)} - 1} - 1\right| \leq rK_U. \tag{IV.9}$$

Increasing K_U if necessary, we also have, $\forall a > 0$,

$$\left| \frac{1}{r^{\frac{1}{\alpha(t+r)} - \frac{1}{\alpha(t)}}} - 1 \right| \leq r^a K_U. \tag{IV.10}$$

For the last term, we write

$$\frac{\int_{\mathbb{R}} f(t+r,t+r,x) f(t,t,x) m(dx)}{\int_{\mathbb{R}} f(t+r,t+r,x)^2 m(dx)} - 1 = \Delta_1(r,t) + \Delta_2(r,t)$$

where

$$\Delta_1(r,t) = \frac{1}{\int_{\mathbb{R}} f(t+r,t+r,x)^2 m(dx)} \left(\int_{\mathbb{R}} f(t+r,t+r,x) \left(f(t,t,x) - f(t+r,t,x) \right) m(dx) \right)$$

and

$$\Delta_2(r,t) = \frac{1}{\int_{\mathbb{R}} f(t+r,t+r,x)^2 m(dx)} \left(\int_{\mathbb{R}} f(t+r,t+r,x) \left(f(t+r,t,x) - f(t+r,t+r,x) \right) m(dx) \right).$$

With **(C13)**, we may choose K_U such that

$$|\Delta_1(r,t)| \leq K_U \int_{\mathbb{R}} |f(t+r,t+r,x)| \, |f(t,t,x) - f(t+r,t,x)| \, m(dx).$$

Let $p \in (\alpha(t), 2)$, $p \geq 1$ satisfying **(C10)**, and q such that $\frac{1}{p} + \frac{1}{q} = 1$. Hölder inequality entails :

$$\begin{aligned} |\Delta_1(r,t)| &\leq K_U \left(\int_{\mathbb{R}} |f(t+r,t+r,x)|^q m(dx) \right)^{1/q} \left(\int_{\mathbb{R}} |f(t,t,x) - f(t+r,t,x)|^p \, m(dx) \right)^{1/p} \\ &\leq K_U \left(\int_{\mathbb{R}} |f(t+r,t,x) - f(t,t,x)|^p \, m(dx) \right)^{1/p} \text{ with } \textbf{(C6)} \text{ and } \textbf{(C12)} \\ &\leq K_U r^{\frac{1}{p} + h(t) - \frac{1}{\alpha(t)}} \text{ with } \textbf{(C10)}. \end{aligned}$$

With **(C12)**, **(C13)** and Cauchy-Schwarz inequality, we select K_U such that

$$\begin{aligned} |\Delta_2(r,t)| &\leq K_U \left(\int_{\mathbb{R}} |f(t+r,t+r,x) - f(t+r,t,x)|^2 \, m(dx) \right)^{\frac{1}{2}} \\ &\leq K_U r \text{ with } \textbf{(C15)}. \end{aligned}$$

Finally, since $h(t) + \frac{1}{p} - \frac{1}{\alpha(t)} \leq 1$,

$$\left| \frac{\int_{\mathbb{R}} f(t+r,t+r,x) f(t,t,x) m(dx)}{\int_{\mathbb{R}} f(t+r,t+r,x)^2 m(dx)} - 1 \right| \leq K_U r^{h(t) + \frac{1}{p} - \frac{1}{\alpha(t)}}. \tag{IV.11}$$

Using the inequalities (IV.7), (IV.8), (IV.9), (IV.10) and (IV.11), we may find a constant K_U such that for all $a > 0$,

$$|\Delta(r,t)| \leq \frac{1}{r^{1+2(h(t)-1/\alpha(t))}} K_U(r^2 + r^{2a} + r^{2(h(t)+\frac{1}{p}-\frac{1}{\alpha(t)})}).$$

Choosing $a \in \left(h(t) + \frac{1}{p} - \frac{1}{\alpha(t)}, 1\right)$, this entails :

$$\begin{aligned} |\Delta(r,t)| &\leq \frac{3}{r^{1+2(h(t)-1/\alpha(t))}} K_U r^{2(h(t)+\frac{1}{p}-\frac{1}{\alpha(t)})} \\ &\leq 3K_U r^{\frac{2}{p}-1}. \end{aligned}$$

Since $\frac{2}{p} - 1 > 0$, $\lim_{r\to 0} |\Delta(r,t)| = 0$ ■

Lemma IV.14. Assuming (C1), (C6), (C7), (C8), one has :

$$\forall \varepsilon < \frac{1}{d}, \exists K_U \leq 1 \text{ such that } \forall v \geq 1, \forall r \leq \varepsilon_0,$$

$$\begin{aligned} y \geq K_U \frac{v^{\frac{d}{1-\varepsilon d}}}{r} \quad &\Rightarrow \quad \sin^2\left(\frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{2r^{h(t)} y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)} f(t,t,x)}{2r^{h(t)} y^{1/\alpha(t)}}\right) \\ &\geq \quad \frac{1}{2}\left|\frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{2r^{h(t)} y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)} f(t,t,x)}{2r^{h(t)} y^{1/\alpha(t)}}\right|^2. \end{aligned}$$

If in addition we suppose (Cu8),

$$\begin{aligned} y \geq K_U \frac{v^{\frac{d}{1-\varepsilon d}}}{r} \quad &\Rightarrow \quad \forall t \in U, \sin^2\left(\frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{2r^{h(t)} y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)} f(t,t,x)}{2r^{h(t)} y^{1/\alpha(t)}}\right) \\ &\geq \quad \frac{1}{2}\left|\frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{2r^{h(t)} y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)} f(t,t,x)}{2r^{h(t)} y^{1/\alpha(t)}}\right|^2. \end{aligned}$$

Proof

Let $\varepsilon < \frac{1}{d}$. We write $\frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{2r^{h(t)} y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)} f(t,t,x)}{2r^{h(t)} y^{1/\alpha(t)}} = \kappa_1(r,t,v,x,y) + \kappa_2(r,t,v,x,y)$, with

$$\kappa_1(r,t,v,x,y) = \frac{v}{2r^{h(t)}}\left(\frac{C_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{y^{1/\alpha(t+r)}} - \frac{C_{\alpha(t)}^{1/\alpha(t)} f(t+r,t,x)}{y^{1/\alpha(t)}}\right)$$

and

$$\kappa_2(r,t,v,x,y) = \frac{vC_{\alpha(t)}^{1/\alpha(t)}}{2r^{h(t)}y^{1/\alpha(t)}}\left(f(t+r,t,x) - f(t,t,x)\right).$$

Using the finite-increments theorem,

$$\begin{aligned}|\kappa_1(r,t,v,x,y)| \quad &\leq \quad \frac{v}{2r^{h(t)}} r(\sup_{a\in U}\left|\frac{K_U|f(t+r,a,x)|}{y^{1/\alpha(a)}}\right| + \sup_{a\in U}\left|\frac{C_{\alpha(a)}^{1/\alpha(a)}|f_v(t+r,a,x)|}{y^{1/\alpha(a)}}\right| \\ &+ \quad \sup_{a\in U}\left|\frac{|\alpha'(a)|}{\alpha^2(a)}|\ln y|\frac{C_{\alpha(a)}^{1/\alpha(a)}|f(t+r,a,x)|}{y^{1/\alpha(a)}}\right|).\end{aligned}$$

For $y \geq 1$, conditions **(C6)** and **(C7)** imply

$$\frac{K_U|f(t+r,a,x)|}{y^{1/\alpha(a)}} \leq \frac{K_U}{y^{1/d}},$$

$$\frac{K_U|f_v(t+r,a,x)|}{y^{1/\alpha(a)}} \leq \frac{K_U}{y^{1/d}},$$

and

$$\frac{|\alpha'(a)|}{\alpha^2(a)}|\ln y|\frac{C_{\alpha(a)}^{1/\alpha(a)}|f(t+r,a,x)|}{y^{1/\alpha(a)}} \leq \frac{K_U|\ln y|}{y^{1/d}}.$$

Finally,

$$\begin{aligned}|\kappa_1(r,t,v,x,y)| \quad &\leq \quad \frac{K_U v r^{1-h(t)}}{y^{1/d}}(1+|\ln y|) \\ &\leq \quad \frac{K_U v}{y^{1/d-\varepsilon}}.\end{aligned}$$

Condition **(C8)** allows to estimate $\kappa_2(r,t,v,x,y)$ as follows :

$$|\kappa_2(r,t,v,x,y)| \leq \frac{K_U v}{(ry)^{1/\alpha(t)}}.$$

Finally, $\forall K > 0$, $\forall \varepsilon < \frac{1}{d}$,$\exists K_U \geq 1$,$\forall v \geq 1$,$\forall r < \varepsilon_0$, $\forall y \geq K_U \frac{v^{\frac{d}{1-\varepsilon d}}}{r}$,

$$\left|\frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{2r^{h(t)}y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)} f(t,t,x)}{2r^{h(t)}y^{1/\alpha(t)}}\right| \leq K \quad \blacksquare$$

Lemma IV.15. Assuming (C6), (C10), (C11), (C12), (C13), (C14), (C15), there exist ε_0 and $K_U > 0$ such that $\forall r < \varepsilon_0$, $\forall v \geq 1$:

$$N(v,t,r) \geq K_U v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{c})},$$

where

$$N(v,t,r) =: \int_{\mathbb{R}} \int_{\frac{K_U v^{\frac{d}{1-\varepsilon d}}}{r}} \left| \frac{v C_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{2r^{h(t)} y^{1/\alpha(t+r)}} - \frac{v C_{\alpha(t)}^{1/\alpha(t)} f(t,t,x)}{2r^{h(t)} y^{1/\alpha(t)}} \right|^2 dy\, m(dx).$$

If in addition we suppose (Cu10), (Cu11), (Cu12), (Cu14), (Cu15), the constant K_U does not depend on t.

Proof

Expanding the square above, we may write

$$N(v,t,r) = A_1(r,t) v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{\alpha(t+r)})} - A_2(r,t) v^{2+\frac{d}{1-\varepsilon d}(1-\frac{1}{\alpha(t+r)}-\frac{1}{\alpha(t)})} + A_3(r,t) v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{\alpha(t)})},$$

where

$$A_1(r,t) = \frac{C_{\alpha(t+r)}^{2/\alpha(t+r)} (K_U)^{1-\frac{2}{\alpha(t+r)}}}{4\left(\frac{2}{\alpha(t+r)}-1\right) r^{1+2(h(t)-\frac{1}{\alpha(t+r)})}} \int_{\mathbb{R}} |f(t+r,t+r,x)|^2 m(dx),$$

$$A_2(r,t) = \frac{C_{\alpha(t+r)}^{1/\alpha(t+r)} C_{\alpha(t)}^{1/\alpha(t)} (K_U)^{1-\frac{1}{\alpha(t+r)}-\frac{1}{\alpha(t)}}}{2\left(\frac{1}{\alpha(t+r)}+\frac{1}{\alpha(t)}-1\right) r^{1+2h(t)-\frac{1}{\alpha(t+r)}-\frac{1}{\alpha(t)}}} \int_{\mathbb{R}} f(t+r,t+r,x) f(t,t,x) m(dx),$$

and

$$A_3(r,t) = \frac{C_{\alpha(t)}^{2/\alpha(t)} (K_U)^{1-\frac{2}{\alpha(t)}}}{4\left(\frac{2}{\alpha(t)}-1\right) r^{1+2(h(t)-\frac{1}{\alpha(t)})}} \int_{\mathbb{R}} |f(t,t,x)|^2 m(dx).$$

We obtain

$$N(v,t,r) = v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{\alpha(t)})} \left(A_1(r,t) (v^{\frac{2d}{1-\varepsilon d}(\frac{1}{\alpha(t)}-\frac{1}{\alpha(t+r)})})^2 - A_2(r,t)(v^{\frac{2d}{1-\varepsilon d}(\frac{1}{\alpha(t)}-\frac{1}{\alpha(t+r)})}) + A_3(r,t) \right).$$

Let $P(r,t,X) = A_1(r,t)X^2 - A_2(r,t)X + A_3(r,t)$. Write :

$$P(r,t,X) = P(r,t,X) - P(r,t,\frac{A_2(r,t)}{2A_1(r,t)}) + P(r,t,\frac{A_2(r,t)}{2A_1(r,t)}) - P(r,t,1) + P(r,t,1).$$

Since $P(\frac{A_2(r,t)}{2A_1(r,t)})$ is the minimum of P,

$$P(r,t,X) \geq P(r,t,\frac{A_2(r,t)}{2A_1(r,t)}) - P(r,t,1) + P(r,t,1).$$

Note that $P(r,t,1) = N(1,t,r)$, thus Lemma (IV.12) entails that there exists a positive function l such that $\lim_{r\to 0} P(r,t,1) = l(t)$. For $P(r,t,\frac{A_2(r,t)}{2A_1(r,t)}) - P(r,t,1)$, we use Lemma (IV.13). With the same notations,

$$\begin{aligned} |P(r,t,\frac{A_2(r,t)}{2A_1(r,t)}) - P(r,t,1)| &= |A_1(r,t)| r^{1+2(h(t)-\frac{1}{\alpha(t)})} \Delta(r,t) \\ &\leq K_U \Delta(r,t) \end{aligned}$$

thus $\lim_{r\to 0} |P(r,t,\frac{A_2(r,t)}{2A_1(r,t)}) - P(r,t,1)| = 0$. As a consequence, there exist a positive constant K_U and $\varepsilon_0 > 0$ such that for all $x \in \mathbb{R}$ and $r \in (0,\varepsilon_0)$, $P(r,t,x) \geq K_U$. We obtain $N(v,t,r) \geq v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{\alpha(t)})}K_U$ for all $v \in \mathbb{R}$ and $r \in (0,\varepsilon_0)$. Since $\alpha(t) > c$, $N(v,t,r) \geq K_U v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{c})}$ ∎

Proof of Theorem IV.1
Consider

$$\mathsf{E}\left[\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^{h(t)}}\right|^\eta\right] = \int_0^\infty \mathsf{P}\left(\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^{h(t)}}\right|^\eta > x\right) dx.$$

Thanks to **(C1)**, **(C2)**, **(C3)** and **(C5)**, Y is $h(t)$-localisable at t [29], thus for all $x > 0$,

$$\mathsf{P}\left(\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^{h(t)}}\right|^\eta > x\right) \to \mathsf{P}\left(\left|Y_t'(1)\right|^\eta > x\right).$$

We shall make use of Lebesgue dominated convergence theorem.

For $x \leq 1$, $\mathsf{P}\left(\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^{h(t)}}\right|^\eta > x\right) \leq 1$.

For $x > 1$,

$$\begin{aligned}\mathsf{P}\left(\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^{h(t)}}\right|^\eta > x\right) &= \mathsf{P}\left(\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^{h(t)}}\right| > x^{1/\eta}\right)\\ &\leq \mathsf{P}\left(\left|\frac{X(t+\varepsilon,t+\varepsilon)-X(t+\varepsilon,t)}{\varepsilon^{h(t)}}\right| > \frac{x^{1/\eta}}{2}\right)\\ &\quad + \mathsf{P}\left(\left|\frac{X(t+\varepsilon,t)-X(t,t)}{\varepsilon^{h(t)}}\right| > \frac{x^{1/\eta}}{2}\right).\end{aligned}$$

For the first term, by Proposition IV.9 (or IV.10),

$$\mathsf{P}\left(\left|\frac{X(t+\varepsilon,t+\varepsilon)-X(t+\varepsilon,t)}{\varepsilon^{h(t)}}\right| > \frac{x^{1/\eta}}{2}\right) \leq \frac{K_U}{x^{d/\eta}}\left(1+|\log x|^d\right) + \frac{K_U}{x^{c/\eta}}\left(1+|\log x|^c\right).$$

For the second term, let $p \in (\eta, \alpha(t))$.

$$\mathsf{P}\left(\left|\frac{X(t+\varepsilon,t)-X(t,t)}{\varepsilon^{h(t)}}\right| > \frac{x^{1/\eta}}{2}\right) = \mathsf{P}\left(\left|\frac{X(t+\varepsilon,t)-X(t,t)}{\varepsilon^{h(t)}}\right|^p > \frac{x^{p/\eta}}{2^p}\right).$$

With Markov inequality and **(C9)**,

$$\begin{aligned}
\mathsf{P}\left(\left|\frac{X(t+\varepsilon,t)-X(t,t)}{\varepsilon^{h(t)}}\right| > \frac{x^{1/\eta}}{2}\right) &\leq \frac{2^p}{x^{p/\eta}\varepsilon^{ph(t)}} C_{\alpha(t),0}(p)^p \|f(t+\varepsilon,t,.)-f(t,t,.)\|^p_{\alpha(t)} \\
&\leq \frac{2^p C_{\alpha(t),0}(p)^p}{x^{p/\eta}\varepsilon^{ph(t)}} \left(\int_{\mathbb{R}} |f(t+\varepsilon,t,x)-f(t,t,x)|^{\alpha(t)} m(dx)\right)^{p/\alpha(t)} \\
&\leq \frac{K_{p,\alpha(t)}}{x^{p/\eta}},
\end{aligned}$$

thus

$$\mathsf{P}(\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^h}\right|^\eta > x) \leq K_U \left(\frac{1}{x^{d/\eta}}\left(1+|\log x|^d\right) + \frac{1}{x^{c/\eta}}\left(1+|\log x|^c\right) + \frac{1}{x^{p/\eta}}\right) \mathbf{1}_{x>1} + \mathbf{1}_{x\leq 1} \blacksquare$$

Proof of Theorem IV.2

Let $\gamma > h(t)$ and $x > 0$.

$$\mathsf{P}\left(\frac{r^\gamma}{|Y(t+r)-Y(t)|} > x\right) = \mathsf{P}\left(|Y(t+r)-Y(t)| < \frac{r^\gamma}{x}\right).$$

Applying Proposition (IV.11), there exists $\varepsilon_0 > 0$ such that

$$\sup_{r\in B(0,\varepsilon_0)} \int_0^{+\infty} \varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v)dv < +\infty.$$

Thus with Proposition (IV.8), there exists $K_U > 0$ such that

$$\mathsf{P}\left(|Y(t+r)-Y(t)| < \frac{r^\gamma}{x}\right) \leq K_U \frac{r^{\gamma-h(t)}}{x}.$$

Let $r_n = \frac{1}{n^\eta}$ with $\eta(\gamma - h(t)) > 1$. $\forall x > 0$, $\sum_n \mathsf{P}\left(\frac{r_n^\gamma}{|Y(t+r_n)-Y(t)|} > x\right) < +\infty$. Borel Cantelli lemma entails that, almost surely, $\lim_{n\to+\infty} \frac{|Y(t+r_n)-Y(t)|}{r_n^\gamma} = +\infty$. As a consequence, almost surely, $\limsup_{r\to 0} \frac{|Y(t+r)-Y(t)|}{r^\gamma} = +\infty$, and

$$\mathcal{H}_t \leq h(t) \qquad \blacksquare$$

Proof of Theorem IV.5

We want to apply Theorem (IV.1) with $f(t,u,x) = \mathbf{1}_{[0,t]}(x)$. Let us show that conditions (C1), (C2), (C3), (C5) and (C9) are satisfied.

- (C1) The family of functions $v \to f(t,v,x)$ is differentiable for all (v,t) in $(0,1)^2$ and almost all x in E. The derivatives of f with respect to u vanish.

- (C2)
$$|f(t,w,x)|^{\alpha(w)} = \mathbf{1}_{[0,t]}(x)$$
thus, for all $\delta > 0$, all $t \in (0,1)$,
$$\int_{\mathbb{R}} \left[\sup_{w\in(0,1)} (|f(t,w,x)|^{\alpha(w)}) \right]^{1+\delta} dx = t$$
and (C2) holds.
- (C3) $f'_u = 0$ thus (C3) holds.
- (C5) $X(t,u)$ (as a process in t) is localisable at u with exponent $\frac{1}{\alpha(u)} \in (\frac{1}{d}, \frac{1}{c}) \subset (0,1)$, with local form $X_u(t,u)$, and $u \mapsto \frac{1}{\alpha(u)}$ is a C^1 function (see [29]).
- (C9)
$$\begin{aligned} \frac{1}{r^{h(t)\alpha(t)}} \int_{\mathbb{R}} |f(t+r,t,x) - f(t,t,x)|^{\alpha(t)} m(dx) &= \frac{1}{r} \int_t^{t+r} dx \\ &= 1, \end{aligned}$$
thus (C9) holds.

From Theorem (IV.1), we get that
$$\mathsf{E}\left[|Y(t+\varepsilon) - Y(t)|^{\eta}\right] \sim \varepsilon^{\frac{\eta}{\alpha(t)}} \mathsf{E}\left[|Y'_t(1)|^{\eta}\right].$$
Since $Y'_t(1)$ is an $S_{\alpha(t)}(1,0,0)$ random variable, Property 1.2.17 of [49] allows to conclude ■

Proof of Theorem IV.6

We want to apply Theorem (IV.2) with $f(t,u,x) = \mathbf{1}_{[0,t]}(x)$ and $h(t) = \frac{1}{\alpha(t)}$ in order to obtain the inequality. Let us show that the conditions (C6), (C7), (Cu8), (Cu10), (Cu11), (Cu12), (C13), (Cu14) and (Cu15) are satisfied.

- (C6) Obvious.
- (C7) Obvious.
- (Cu8) $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbb{R}$,
$$\begin{aligned} \frac{1}{|v-u|^{h(u)-1/\alpha(u)}} |f(v,u,x) - f(u,u,x)| &= \mathbf{1}_{[u,v]}(x) \\ &\leq 1 \end{aligned}$$
thus (Cu8) holds.
- (Cu10) $\forall v \in U$,$\forall u \in U$,
$$\begin{aligned} \frac{1}{|v-u|^{1+p(h(u)-\frac{1}{\alpha(u)})}} \int_{\mathbb{R}} |f(v,u,x) - f(u,u,x)|^p \, m(dx) &= \frac{1}{|v-u|} \int_{\mathbb{R}} |\mathbf{1}_{[u,v]}(x)| \\ &= 1 \end{aligned}$$
thus (Cu10) holds.

- (Cu11) $\forall v \in U$, $\forall u \in U$,

$$\int_{\mathbb{R}} |f(v,u,x)|^2 \, m(dx) = v$$

thus (Cu11) holds ($U = (0,1)$).
- (Cu12) For the same reason as (Cu11), (Cu12) holds.
- (C13) Since $t \in (0,1)$ (in particular $t \neq 0$), one can choose U such that $\inf_{v \in U} v > 0$ thus (C13) holds.
- (Cu14)

$$\begin{aligned} \frac{1}{r^{1+2(h(t)-1/\alpha(t))}} \int_{\mathbb{R}} (f(t+r,t,x) - f(t,t,x))^2 \, m(dx) &= \frac{1}{r} \int_{\mathbb{R}} \mathbf{1}_{[t,t+r]}(x) dx \\ &= 1 \end{aligned}$$

thus (Cu14) holds.
- (Cu15) $\forall v \in U$, $\forall u \in U$,

$$\frac{1}{|v-u|^2} \int_{\mathbb{R}} |f(v,v,x) - f(v,u,x)|^2 \, m(dx) = 0$$

thus (Cu15) holds ■

Proof of Theorem IV.3

We want to apply Theorem (IV.1) with $f(t,u,x) = |t-x|^{H(u)-\frac{1}{\alpha(u)}} - |x|^{H(u)-\frac{1}{\alpha(u)}}$. Let us show that conditions (C1), (Cs2), (Cs3), (Cs4), (C5) and (C9) are satisfied.

- (C1) The family of functions $u \to f(t,u,x)$ is differentiable for all (u,t) in a neighbourhood of t_0 and almost all x in E. The derivatives of f with respect to u read :

$$f'_u(t,w,x) = (h'(w) + \frac{\alpha'(w)}{\alpha^2(w)}) \left[(\log|t-x|)|t-x|^{h(w)-1/\alpha(w)} - (\log|x|)|x|^{h(w)-1/\alpha(w)} \right].$$

- (Cs2) In [16], it is shown that, given $t_0 \in \mathbb{R}$, one may choose $\varepsilon > 0$ small enough and numbers a, b, h_-, h_+ with $0 < a < \alpha(w) < b < 2$, $0 < h_- < h(w) < h_+ < 1$ and $\frac{a}{b}(\frac{1}{a} - \frac{1}{b}) < h_- - (\frac{1}{a} - \frac{1}{b}) < h_- < h_+ < h_+ + (\frac{1}{a} - \frac{1}{b}) < 1 - (\frac{1}{a} - \frac{1}{b})$ such that, for all t and w in $U := (t_0 - \varepsilon, t_0 + \varepsilon)$ and all real x :

$$|f(t,w,x)|, |f'_{t_0}(t,w,x)| \leq k_1(t,x) \tag{IV.12}$$

where

$$k_1(t,x) = \begin{cases} c_1 \max\{1, |t-x|^{h_- - 1/a} + |x|^{h_- - 1/a}\} & (|x| \leq 1 + 2\max_{t \in U} |t|) \\ c_2 |x|^{h_+ - 1/b - 1} & (|x| > 1 + 2\max_{t \in U} |t|) \end{cases} \tag{IV.13}$$

for appropriately chosen constants c_1 and c_2. One has, for all $\delta > 0$,

$$\begin{aligned}
\int_{\mathbb{R}} \left[\sup_{w\in U} |f(t,w,x)|^{\alpha(w)}\right]^{1+\delta} r(x)^\delta dx &\leq \int_{\mathbb{R}} \left(k_1(t,x)^a + k_1(t,x)^b\right)^{1+\delta} r(x)^\delta dx \\
&\leq K_\delta \int_{\mathbb{R}} k_1(t,x)^{a(1+\delta)} r(x)^\delta dx \\
&\quad + K_\delta \int_{\mathbb{R}} k_1(t,x)^{b(1+\delta)} r(x)^\delta dx.
\end{aligned}$$

Let us study $\int_{\mathbb{R}} k_1(t,x)^{p(1+\delta)} r(x)^\delta dx$, where $p = a$ or $p = b$.

$$\begin{aligned}
\int_{\mathbb{R}} k_1(t,x)^{p(1+\delta)} r(x)^\delta dx &= \frac{\pi^{2\delta}}{3^\delta} \sum_{j=0}^{+\infty} (j+1)^{2\delta} \int_j^{j+1} (k_1(t,-x)^{p(1+\delta)} + k_1(t,x)^{p(1+\delta)}) dx \\
&= \frac{\pi^{2\delta}}{3^\delta} \sum_{j=0}^{+\infty} (j+1)^{2\delta} \int_j^{j+1} (k_1(-t,x)^{p(1+\delta)} + k_1(t,x)^{p(1+\delta)}) dx.
\end{aligned}$$

We consider now $\int_j^{j+1} k_1(\pm t,x)^{p(1+\delta)} dx$. There exists $K_{p,\delta} > 0$ such that, for all real x such that $|x| \leq 1 + 2\max_{t\in U}|t|$:

$$k_1(\pm t,x)^{p(1+\delta)} \leq K_{p,\delta}\left(1 + |\pm t - x|^{p(1+\delta)(h_- - 1/a)} + |x|^{p(1+\delta)(h_- - 1/a)}\right),$$

and for all real x such that $|x| > 1 + 2\max_{t\in U}|t|$,

$$k_1(\pm t,x)^{p(1+\delta)} \leq K_{p,\delta}|x|^{p(1+\delta)(h_+ - 1/b - 1)}.$$

Let $j_0 = [1 + 2\max_{t\in U}|t|)]$. For $j < j_0$,

$$\int_j^{j+1} k_1(\pm t,x)^{p(1+\delta)} dx \quad \leq \quad K_{p,\delta}(1 + \int_j^{j+1} |\pm t - x|^{p(1+\delta)(h_- - 1/a)} dx + \int_j^{j+1} |x|^{p(1+\delta)(h_- - 1/a)} dx).$$

Choose δ such that $p(1+\delta)(h_- - 1/a) > -1$ (we show below that such a δ exists). Then

$$\begin{aligned}
\int_j^{j+1} |\pm t - x|^{p(1+\delta)(h_- - 1/a)} dx &= \int_{\pm t - j - 1}^{\pm t - j} |u|^{p(1+\delta)(h_- - 1/a)} du \\
&\leq \frac{|\pm t - j|^{1+p(1+\delta)(h_- - 1/a)} + |\pm t - j - 1|^{1+p(1+\delta)(h_- - 1/a)}}{1 + p(1+\delta)(h_- - 1/a)} \\
&\leq K_U |t|^{1+p(1+\delta)(h_- - 1/a)} |1 + j|^{1+p(1+\delta)(h_- - 1/a)} \\
&\leq K_U (1+j)^{1+p(1+\delta)(h_- - 1/a)}
\end{aligned}$$

where $K_U > 0$ depends on U and may have changed from line to line. We deduce :

$$\int_j^{j+1} k_1(\pm t, x)^{p(1+\delta)} dx \leq K_U (1 + j^{1+p(1+\delta)(h_- - 1/a)}).$$

For $j = j_0$,

$$\begin{aligned}\int_{j_0}^{j_0+1} k_1(\pm t, x)^{p(1+\delta)} dx &\leq K_U |j_0|^{1+p(1+\delta)(h_- - 1/a)} + K_U \int_{j_0}^{j_0+1} |x|^{p(1+\delta)(h_+ - 1/b - 1)} dx \\ &\leq K_U.\end{aligned}$$

For $j > j_0$,

$$\begin{aligned}\int_j^{j+1} k_1(\pm t, x)^{p(1+\delta)} dx &\leq K_U \int_j^{j+1} |x|^{p(1+\delta)(h_+ - 1/b - 1)} dx \\ &\leq K_U j^{p(1+\delta)(h_+ - 1/b - 1)}.\end{aligned}$$

Finally,

$$\begin{aligned}\sup_{t \in U} \int_{\mathbb{R}} k_1(t, x)^{p(1+\delta)} r(x)^\delta dx &\leq K_U \left(1 + \sum_{j=0}^{j_0-1} j^{2\delta}(1 + j^{1+p(1+\delta)(h_- - 1/a)})\right) \\ &\quad + K_U \sum_{j=j_0+1}^{\infty} j^{2\delta + p(1+\delta)(h_+ - 1/b - 1)}.\end{aligned}$$

To conclude, we need to show that we may chose $\delta > \frac{b}{a} - 1$ such that $p(1+\delta)(h_- - 1/a) > -1$ and $2\delta + p(1+\delta)(h_+ - 1/b - 1) < -1$, for $p = a$ and $p = b$. We consider several cases.
<u>First case</u> : $h_- - \frac{1}{a} \geq 0$ and $h_+ - \frac{1}{b} - 1 \leq -\frac{2}{a}$.
Let $\delta > \frac{b}{a} - 1$. One has $p(1+\delta)(h_- - \frac{1}{a}) \geq 0 > -1$. We consider $1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1)$.

$$\begin{aligned}1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) &\leq 1 + 2\delta - \frac{2}{a} p(1+\delta) \\ &= 1 - \frac{2p}{a} + 2\delta(1 - \frac{p}{a}).\end{aligned}$$

Since $1 - \frac{2p}{a} < 0$ and $1 - \frac{p}{a} \leq 0$, $1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) < 0$.
<u>Second case</u> : $h_- - \frac{1}{a} \geq 0$ and $h_+ - \frac{1}{b} - 1 > -\frac{2}{a}$.
Let $\delta \in \left(\frac{b}{a} - 1, \frac{\frac{1}{b} - \frac{1}{a} + 1 - h_+}{\frac{2}{a} - \frac{1}{b} - 1 + h_+}\right)$. One has $p(1+\delta)(h_- - \frac{1}{a}) \geq 0 > -1$. We consider $1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1)$.

For $p = a$:

$$\begin{aligned}
1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) &= a\delta(\frac{2}{a} + h_+ - \frac{1}{b} - 1) + a(h_+ - \frac{1}{b} - 1 + \frac{1}{a}) \\
&< a(\frac{1}{b} - \frac{1}{a} + 1 - h_+) + a(h_+ - \frac{1}{b} - 1 + \frac{1}{a}) \\
&= 0.
\end{aligned}$$

For $p = b$:

$$1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) = b\delta(\frac{1}{b} + h_+ - 1) + b(h_+ - 1).$$

If $\frac{1}{b} + h_+ - 1 \leq 0$, then $b\delta(\frac{1}{b} + h_+ - 1) + b(h_+ - 1) < 0$. Else

$$\begin{aligned}
b\delta(\frac{1}{b} + h_+ - 1) + b(h_+ - 1) &< b\frac{\frac{1}{b} - \frac{1}{a} + 1 - h_+}{\frac{2}{a} - \frac{1}{b} - 1 + h_+}(\frac{1}{b} + h_+ - 1) + b(h_+ - 1) \\
&= \frac{b}{\frac{2}{a} - \frac{1}{b} - 1 + h_+}(\frac{1}{a} - \frac{1}{b})(h_+ - 1 - \frac{1}{b}) \\
&< 0.
\end{aligned}$$

Third case : $h_- - \frac{1}{a} < 0$ and $h_+ - \frac{1}{b} - 1 \leq -\frac{2}{a}$.
Let $\delta \in \left(\frac{b}{a} - 1, \frac{ah_- + \frac{a}{b} - 1}{1 - ah_-}\right)$.
For $p = a$:

$$\begin{aligned}
1 + p(1+\delta)(h_- - \frac{1}{a}) &= ah_- + \delta(ah_- - 1) \\
&> ah_- + (ah_- - 1)\frac{ah_- + \frac{a}{b} - 1}{1 - ah_-} \\
&= ah_- + 1 - \frac{a}{b} - ah_- \\
&> 0,
\end{aligned}$$

and

$$\begin{aligned}
1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) &= a\delta(\frac{2}{a} + h_+ - \frac{1}{b} - 1) + a(h_+ - \frac{1}{b} - 1 + \frac{1}{a}) \\
&\leq a(h_+ - \frac{1}{b} - 1 + \frac{1}{a}) \\
&\leq -1 \\
&< 0.
\end{aligned}$$

For $p = b$:

$$\begin{aligned} 1+p(1+\delta)(h_- - \frac{1}{a}) &= b(h_- - \frac{1}{a} + \frac{1}{b}) + b\delta(h_- - \frac{1}{a}) \\ &> b(h_- - \frac{1}{a} + \frac{1}{b}) + b(h_- - \frac{1}{a})\frac{ah_- + \frac{a}{b} - 1}{1 - ah_-} \\ &= b(h_- - \frac{1}{a} + \frac{1}{b}) + b(\frac{1}{a} - \frac{1}{b} - h_-) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} 1+2\delta+p(1+\delta)(h_+ - 1/b - 1) &= b\delta(\frac{1}{b} + h_+ - 1) + b(h_+ - 1) \\ &\leq b\delta(\frac{2}{b} - \frac{2}{a}) + b(h_+ - 1) \\ &< 0. \end{aligned}$$

Fourth case : $h_- - \frac{1}{a} < 0$ and $h_+ - \frac{1}{b} - 1 > -\frac{2}{a}$.
Let $\delta \in \left(\frac{b}{a} - 1, \min(\frac{ah_- + \frac{a}{b} - 1}{1 - ah_-}, \frac{\frac{1}{b} - \frac{1}{a} + 1 - h_+}{\frac{2}{a} - \frac{1}{b} - 1 + h_+})\right)$.
For $p = a$:

$$\begin{aligned} 1+p(1+\delta)(h_- - \frac{1}{a}) &= ah_- + \delta(ah_- - 1) \\ &> ah_- + (ah_- - 1)\frac{ah_- + \frac{a}{b} - 1}{1 - ah_-} \\ &= ah_- + 1 - \frac{a}{b} - ah_- \\ &> 0, \end{aligned}$$

and

$$\begin{aligned} 1+2\delta+p(1+\delta)(h_+ - 1/b - 1) &= a\delta(\frac{2}{a} + h_+ - \frac{1}{b} - 1) + a(h_+ - \frac{1}{b} - 1 + \frac{1}{a}) \\ &> a(\frac{1}{b} - \frac{1}{a} + 1 - h_+) + a(h_+ - \frac{1}{b} - 1 + \frac{1}{a}) \\ &= 0. \end{aligned}$$

For $p = b$:

$$\begin{aligned} 1+p(1+\delta)(h_- - \frac{1}{a}) &= b(h_- - \frac{1}{a} + \frac{1}{b}) + b\delta(h_- - \frac{1}{a}) \\ &> b(h_- - \frac{1}{a} + \frac{1}{b}) + b(h_- - \frac{1}{a})\frac{ah_- + \frac{a}{b} - 1}{1 - ah_-} \\ &= b(h_- - \frac{1}{a} + \frac{1}{b}) + b(\frac{1}{a} - \frac{1}{b} - h_-) \\ &= 0, \end{aligned}$$

and

$$1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) = b\delta(\frac{1}{b} + h_+ - 1) + b(h_+ - 1).$$

If $\frac{1}{b} + h_+ - 1 \leq 0$, then $1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) < 0$, else

$$\begin{aligned} b\delta(\frac{1}{b} + h_+ - 1) + b(h_+ - 1) &< b(\frac{\frac{1}{b} - \frac{1}{a} + 1 - h_+}{\frac{2}{a} - \frac{1}{b} - 1 + h_+})(\frac{1}{b} + h_+ - 1) + b(h_+ - 1) \\ &= \frac{b}{\frac{2}{a} - \frac{1}{b} - 1 + h_+}(\frac{1}{a} - \frac{1}{b})(h_+ - 1 - \frac{1}{b}) \\ &< 0. \end{aligned}$$

- (Cs3) is obtained with (IV.12) for the same reason as in (Cs2).
- (Cs4) For j large enough ($j > j^*$),

$$\begin{aligned} |f(t,w,x)\log(r(x))|^{\alpha(w)} &\leq K_1|k_1(t,x)|^{\alpha(w)} \\ &\quad + K_2\sum_{j=j^*}^{+\infty}|f(t,w,x)|^{\alpha(w)}(\log(j))^d \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x). \end{aligned}$$

$$|f(t,w,x)|^{\alpha(w)}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x) \leq K_2\frac{1}{|x|^{a(1+1/b-h_+)}}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)$$

(K_2 may have changed from line to line). Thus

$$\begin{aligned} \left[\sup_{w\in U}\left[|f(t,w,x)\log(r(x))|^{\alpha(w)}\right]\right]^{1+\delta} r(x)^\delta &\leq K|k_1(t,x)|^{a(1+\delta)}r(x)^\delta + K|k_1(t,x)|^{b(1+\delta)}r(x)^\delta \\ &\quad + K\sum_{j=j^*}^{+\infty}\frac{j^{2\delta}(\log(j))^d}{|x|^{a(1+\delta)(1+1/b-h_+)}}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x). \end{aligned}$$

Let $\delta > \frac{b}{a} - 1$ be such that (Cs2) holds. Since $2\delta + a(1+\delta)(h_+ - 1 - \frac{1}{b}) < -1$, (Cs4) holds .

- (C5) $X(t,u)$ (as a process in t) is localisable at u with exponent $H(u) \in (h_-, h_+) \subset (0,1)$, with local form $X_u(t,u)$, and $u \mapsto H(u)$ is a C^1 function (see [29]).
- (C9)

$$\frac{1}{r^{H(t)\alpha(t)}} \int_{\mathbb{R}} |f(t+r,t,x) - f(t,t,x)|^{\alpha(t)} m(dx) = \int_{\mathbb{R}} \left| |1-x|^{H(t)-\frac{1}{\alpha(t)}} - |x|^{H(t)-\frac{1}{\alpha(t)}} \right|^{\alpha(t)} dx$$

so (C9) holds.

From Theorem IV.1, we obtain that

$$\mathsf{E}\left[|Y(t+\varepsilon) - Y(t)|^{\eta}\right] \sim \varepsilon^{\eta H(t)} \mathsf{E}\left[|Y'_t(1)|^{\eta}\right].$$

Since $Y'_t(1)$ is an $S_{\alpha(t)}(\sigma, 0, 0)$ random variable with

$$\sigma = \left(\int_{\mathbb{R}} \left| |1-x|^{H(t)-\frac{1}{\alpha(t)}} - |x|^{H(t)-\frac{1}{\alpha(t)}} \right|^{\alpha(t)} dx \right)^{\frac{1}{\alpha(t)}},$$

Property 1.2.17 of [49] allows to conclude ■

Proof of Theorem IV.4

We want to apply Theorems IV.2 with $f(t,u,x) = |t-x|^{H(u)-\frac{1}{\alpha(u)}} - |x|^{H(u)-\frac{1}{\alpha(u)}}$ in order to obtain the inequality. Let us show that conditions (C6), (C7), (Cu8), (Cu10), (Cu11), (Cu12), (C13), (Cu14) and (Cu15) are satisfied.

- (C6) Since $H(t) - \frac{1}{\alpha(t)} \geq 0$, (C6) holds.
- (C7) We also use the fact that $H(t) - \frac{1}{\alpha(t)} \geq 0$ in order to prove that (C7) holds.
- (Cu8) $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbb{R}$,

$$\begin{aligned} \frac{1}{|v-u|^{H(u)-1/\alpha(u)}} |f(v,u,x) - f(u,u,x)| &= \frac{1}{|v-u|^{H(u)-1/\alpha(u)}} \left| |v-x|^{H(u)-\frac{1}{\alpha(u)}} - |u-x|^{H(u)-\frac{1}{\alpha(u)}} \right| \\ &\leq 1 \end{aligned}$$

thus (Cu8) holds.
- (Cu10) $\forall v \in U$,$\forall u \in U$,

$$\frac{1}{|v-u|^{1+p(H(u)-\frac{1}{\alpha(u)})}} \int_{\mathbb{R}} |f(v,u,x) - f(u,u,x)|^p m(dx) = \int_{\mathbb{R}} \left| |1-x|^{H(u)-\frac{1}{\alpha(u)}} - |x|^{H(u)-\frac{1}{\alpha(u)}} \right|^p dx$$

so (Cu10) holds.
- (Cu11) $\forall v \in U$, $\forall u \in U$,

$$\int_{\mathbb{R}} |f(v,u,x)|^2 m(dx) = v^{1+2(H(u)-\frac{1}{\alpha(u)})} \int_{\mathbb{R}} \left| |1-x|^{H(u)-\frac{1}{\alpha(u)}} - |x|^{H(u)-\frac{1}{\alpha(u)}} \right|^2 dx$$

thus (Cu11) holds.

- (Cu12) For the same reason as (Cu11), (Cu12) holds.
- (C13) For $t \neq 0$, one can choose U such that $\inf_{v \in U} v^{1+2(H(v)-\frac{1}{\alpha(v)})} > 0$ thus (C13) holds.
- (Cu14)

$$\frac{1}{r^{1+2(H(t)-1/\alpha(t))}} \int_{\mathbb{R}} (f(t+r,t,x) - f(t,t,x))^2 \, m(dx) \quad = \quad \int_{\mathbb{R}} \left| |1-x|^{H(t)-\frac{1}{\alpha(t)}} - |x|^{H(t)-\frac{1}{\alpha(t)}} \right|^2 dx$$

thus, choosing $g(t) = \int_{\mathbb{R}} \left| |1-x|^{H(t)-\frac{1}{\alpha(t)}} - |x|^{H(t)-\frac{1}{\alpha(t)}} \right|^2 dx$, (Cu14) holds.
- (Cu15) $\forall v \in U$, $\forall u \in U$,

$$\frac{1}{|v-u|^2} \int_{\mathbb{R}} |f(v,v,x) - f(v,u,x)|^2 \, m(dx) =$$
$$\frac{1}{|v-u|^2} \int_{\mathbb{R}} \left| |v-x|^{H(v)-\frac{1}{\alpha(v)}} - |v-x|^{H(u)-\frac{1}{\alpha(u)}} - |x|^{H(v)-\frac{1}{\alpha(v)}} + |x|^{H(u)-\frac{1}{\alpha(u)}} \right|^2 dx$$

thus (Cu15) holds ■

IV.4 Proof of Theorem IV.7

Recall the definition of the Lévy Multistable field on $[0,1]$:

$$X(v,u) = C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} \mathbf{1}_{[0,v]}(V_i).$$

To prove Theorem IV.7, we need a series of lemma :

Lemma IV.16. Assume α is $\mathcal{C}^1$. Then, for all $u \in (0,1)$, almost surely,

$$\sup_{v \in [0,1]} \frac{|X(v,v) - X(v,u)|}{|v-u|} < +\infty.$$

Proof
In the case of the Lévy multistable field, (IV.4) reads :

$$X(v,v) - X(v,u) = (v-u) \left(\sum_{i=1}^{+\infty} Z_i^1(v) + \sum_{i=1}^{+\infty} Z_i^3(v) + \sum_{i=1}^{+\infty} Y_i^1(v) + \sum_{i=1}^{+\infty} Y_i^3(v) \right),$$

where $Z_i^1, \ldots$ are defined as above. Let $A > 0$ and $B > 0$ be constants such that $\forall w \in U$, $|a'(w)| \leq A$ and $|a(w)\frac{\alpha'(w)}{\alpha^2(w)}| \leq B$. We write $\sum_{i=1}^{+\infty} Z_i^1(v) = \sum_{j=1}^{+\infty} \left(\sum_{i=2^j}^{2^{j+1}-1} Z_i^1(v) \right) =: \sum_{j=1}^{+\infty} X_j^1(v)$ and $\sum_{i=1}^{+\infty} Z_i^3(v) = \sum_{j=1}^{+\infty} \left(\sum_{i=2^j}^{2^{j+1}-1} Z_i^3(v) \right) =: \sum_{j=1}^{+\infty} X_j^3(v)$. We consider $\liminf_j \{ \sup_{v \in [0,1]} |X_j^1(v)| \leq \frac{Aj\sqrt{2^j}}{2^{j/d}} \}$

and $\liminf_j\{\sup_{v\in[0,1]}|X_j^3(v)|\leq\frac{\log(2)Bj(j+1)\sqrt{2^j}}{2^{j/d}}\}$. Let $V^{(1)},V^{(2)},...,V^{(2^j)}$ denote the order statistics of the V_i ($i.e. V^{(1)}=\min V_i,\ \ldots$). Then :

$$\{\sup_{v\in[0,1]}|X_j^1(v)|>\frac{Aj\sqrt{2^j}}{2^{j/d}}\}\subset\cup_{N\geq 1}^{2^j}\cup_{l_1,\ldots,l_N\in[\![2^j,2^{j+1}-1]\!]}\left(\{|\sum_{i=1}^N\gamma_{l_i}a'(w_{l_i})l_i^{-1/\alpha(w_{l_i})}|>\frac{Aj\sqrt{2^j}}{2^{j/d}}\}\ ..\right.$$

$$\left... \cap\{V^{(1)}=V_{l_1},V^{(2)}=V_{l_2},...,V^{(N)}=V_{l_N}\}\right).$$

$$\begin{aligned}\mathsf{P}\left(\sup_{v\in[0,1]}|X_j^1(v)|>\frac{Aj\sqrt{2^j}}{2^{j/d}}\right) &\leq \sum_{N=1}^{2^j}\sum_{l_1,\ldots,l_N\in[\![2^j,2^{j+1}-1]\!]}\frac{(2^j-N)!}{(2^j)!}\mathsf{P}\left(|\sum_{i=1}^N\gamma_{l_i}a'(w_{l_i})l_i^{-1/\alpha(w_{l_i})}|>\frac{Aj\sqrt{2^j}}{2^{j/d}}\right)\\
&\leq \sum_{N=1}^{2^j}\sum_{l_1,\ldots,l_N\in[\![2^j,2^{j+1}-1]\!]}\frac{(2^j-N)!}{(2^j)!}\mathsf{P}\left(|\sum_{i=1}^N\gamma_{l_i}\frac{a'(w_{l_i})}{A}\frac{2^{j/d}}{l_i^{1/\alpha(w_{l_i})}}|>j\sqrt{N}\right)\\
&\leq \sum_{N=1}^{2^j}\sum_{l_1,\ldots,l_N\in[\![2^j,2^{j+1}-1]\!]}\frac{(2^j-N)!}{(2^j)!}2e^{-\frac{j^2}{2}}\\
&\leq 2e^{-\frac{j^2}{2}}\sum_{N=1}^{2^j}\frac{1}{N!}\\
&\leq 2e^{1-\frac{j^2}{2}}\end{aligned}$$

where we have used the following inequality (Lemma 1.5, chapter 1 in [34]) :

$$\mathsf{P}\left(|\sum_{i=1}^n u_i|\geq\lambda\sqrt{n}\right)\leq 2e^{-\frac{\lambda^2}{2}}$$

for $(u_i)_i$ independent centered random variables verifying $-1\leq u_i\leq 1$, with $u_i=\gamma_{l_i}\frac{a'(w_{l_i})}{A}\frac{2^{j/d}}{l_i^{1/\alpha(w_{l_i})}}$ and $\lambda=j$.

We deduce that $\mathsf{P}\left(\liminf_j\left\{\sup_{v\in[0,1]}|X_j^1(v)|\leq\frac{Aj\sqrt{2^j}}{2^{j/d}}\right\}\right)=1$.

Similarly :

$$\mathsf{P}\left(\sup_{v\in[0,1]}|X_j^3(v)|>\frac{\log(2)Bj(j+1)\sqrt{2^j}}{2^{j/d}}\right)\quad\leq\quad 2e^{1-\frac{j^2}{2}}$$

and $\mathsf{P}\left(\liminf_j\left\{\sup_{v\in[0,1]}|X_j^3(v)|\leq\frac{\log(2)Bj(j+1)\sqrt{2^j}}{2^{j/d}}\right\}\right)=1$. We work on the event

$$\liminf_j\left\{\sup_{v\in[0,1]}|X_j^1(v)|\leq\frac{Aj\sqrt{2^j}}{2^{j/d}}\right\}\cap\liminf_j\left\{\sup_{v\in[0,1]}|X_j^3(v)|\leq\frac{\log(2)Bj(j+1)\sqrt{2^j}}{2^{j/d}}\right\}\cap\liminf_i\{\Gamma_i>1\}.$$

There exists $J_0 \in \mathbb{N}$ such that $\forall j \geq J_0$, $\sup_{v\in[0,1]} |X_j^1(v)| \leq \frac{Aj\sqrt{2^j}}{2^{j/d}}$ and $\sup_{v\in[0,1]} |X_j^3(v)| \leq \frac{\log(2)Bj(j+1)\sqrt{2^j}}{2^{j/d}}$.

$$\left|\sum_{i=1}^{+\infty} Z_i^1(v)\right| \leq \sum_{j=0}^{2^{J_0}-1} \frac{A}{i^{1/d}} + \sum_{j=J_0}^{+\infty} A \frac{j}{2^{j(\frac{1}{d}-\frac{1}{2})}}$$

and

$$\left|\sum_{i=1}^{+\infty} Z_i^3(v)\right| \leq \sum_{j=0}^{2^{J_0}-1} \frac{B\log(i)}{i^{1/d}} + \sum_{j=J_0}^{+\infty} B\log(2) \frac{j(j+1)}{2^{j(\frac{1}{d}-\frac{1}{2})}},$$

thus $\sup_{v\in[0,1]} \left|\sum_{i=1}^{+\infty} Z_i^1(v)\right| < +\infty$ and $\sup_{v\in[0,1]} \left|\sum_{i=1}^{+\infty} Z_i^3(v)\right| < +\infty$.

Fix $i_0 \in \mathbb{N}$ such that $\forall i \geq i_0$, $\Gamma_i > 1$.

$$|\sum_{i=1}^{i_0} Y_i^1(v)| \leq A \sum_{i=1}^{i_0} (\frac{1}{\Gamma_i^{1/c}} + \frac{1}{i^{1/d}})$$

and

$$|\sum_{i=1}^{i_0} Y_i^3(v)| \leq B \sum_{i=1}^{i_0} \left(|\frac{\log \Gamma_i}{\Gamma_i^{1/c}}| + \frac{\log(i)}{i^{1/d}} \right).$$

$$\begin{aligned}
|\sum_{i=i_0}^{+\infty} Y_i^1(v)| \quad \leq \quad & A \sum_{i=i_0}^{+\infty} |\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}| \mathbf{1}_{\{1<\Gamma_i\leq \frac{i}{2}\}} \\
& + A \sum_{i=i_0}^{+\infty} |\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}| \mathbf{1}_{\{\frac{i}{2}<\Gamma_i\leq 2i\}} \\
& + A \sum_{i=i_0}^{+\infty} |\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}| \mathbf{1}_{\{\Gamma_i>2i\}},
\end{aligned}$$

$$\begin{aligned}
|\sum_{i=i_0}^{+\infty} Y_i^1(v)| \quad & \leq \quad 2A \sum_{i=i_0}^{+\infty} (\mathbf{1}_{\{1<\Gamma_i\leq \frac{i}{2}\}} + \mathbf{1}_{\{\Gamma_i>2i\}}) + A \sum_{i=i_0}^{+\infty} |\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}| \mathbf{1}_{\{\frac{i}{2}<\Gamma_i\leq 2i\}} \\
& \leq \quad 2A \sum_{i=i_0}^{+\infty} (\mathbf{1}_{\{1<\Gamma_i\leq \frac{i}{2}\}} + \mathbf{1}_{\{\Gamma_i>2i\}}) + K_{c,d} \sum_{i=i_0}^{+\infty} \frac{1}{i^{\frac{1}{d}}} |\frac{\Gamma_i}{i} - 1|.
\end{aligned}$$

$$
\begin{aligned}
|\sum_{i=i_0}^{+\infty} Y_i^3(v)| \quad \leq \quad & B\sum_{i=i_0}^{+\infty} |\log(\Gamma_i)\Gamma_i^{-1/\alpha(x_i)} - \log(i)i^{-1/\alpha(x_i)}|\mathbf{1}_{\{1<\Gamma_i\leq \frac{i}{2}\}} \\
& +B\sum_{i=i_0}^{+\infty} |\log(\Gamma_i)\Gamma_i^{-1/\alpha(x_i)} - \log(i)i^{-1/\alpha(x_i)}|\mathbf{1}_{\{\frac{i}{2}<\Gamma_i\leq 2i\}} \\
& +B\sum_{i=i_0}^{+\infty} |\log(\Gamma_i)\Gamma_i^{-1/\alpha(x_i)} - \log(i)i^{-1/\alpha(x_i)}|\mathbf{1}_{\{\Gamma_i> 2i\}},
\end{aligned}
$$

$$
\begin{aligned}
|\sum_{i=i_0}^{+\infty} Y_i^3(v)| \quad \leq \quad & K\sum_{i=i_0}^{+\infty} \log(i)(\mathbf{1}_{\{1<\Gamma_i\leq \frac{i}{2}\}} + \mathbf{1}_{\{\Gamma_i>2i\}}) \\
& +B\sum_{i=i_0}^{+\infty} |\log(\Gamma_i)\Gamma_i^{-1/\alpha(x_i)} - \log(i)i^{-1/\alpha(x_i)}|\mathbf{1}_{\{\frac{i}{2}<\Gamma_i\leq 2i\}} \\
\leq \quad & K\sum_{i=i_0}^{+\infty} \log(i)(\mathbf{1}_{\{1<\Gamma_i\leq \frac{i}{2}\}} + \mathbf{1}_{\{\Gamma_i>2i\}}) + K_{c,d}\sum_{i=i_0}^{+\infty} \frac{\log(i)}{i^{\frac{1}{d}}}|\frac{\Gamma_i}{i} - 1|.
\end{aligned}
$$

Finally, $\sup_{v\in[0,1]}\left|\sum_{i=1}^{+\infty} Y_i^1(v)\right| < +\infty$ and $\sup_{v\in[0,1]}\left|\sum_{i=1}^{+\infty} Y_i^3(v)\right| < +\infty$.

As a consequence, $\sup_{v\in[0,1]} \frac{|X(v,v)-X(v,u)|}{|v-u|} < +\infty$ ■

Lemma IV.17. For all $u \in (0,1)$ and all $\eta \in (0, \frac{1}{\alpha(u)})$, one has, almost surely,

$$
\sup_{v\in[0,1]}\left|\frac{X(v,u)-X(u,u)}{|v-u|^\eta}\right| < +\infty.
$$

Proof

Let $\eta \in (0, \frac{1}{\alpha(u)})$, $m \in \mathbb{N}$, $C_j = \cap_{i=2^j}^{2^{j+1}-1}\{V_i \notin [u - \frac{1}{j^2 2^j}, u + \frac{1}{j^2 2^j}]\}$,

$$
D_j^m = \left\{ \sup_{\frac{1}{2^{m+1}}\leq|v-u|\leq\frac{1}{2^m}} \left|\sum_{i=2^j}^{2^{j+1}-1} \gamma_i i^{-1/\alpha(u)} \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta}\right| \leq \frac{1}{j^2} \right\},
$$

and $D_j = \cap_{m\geq 0} D_j^m$. D_j may be written :

$$
D_j = \left\{ \sup_{v\in[0,1]} \left|\sum_{i=2^j}^{2^{j+1}-1} \gamma_i i^{-1/\alpha(u)} \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta}\right| \leq \frac{1}{j^2} \right\}.
$$

Let us evaluate $\liminf C_j$.

$$\mathsf{P}\left(\overline{C_j}\right) \leq \sum_{i=2^j}^{2^{j+1}-1} \frac{1}{j^2 2^j} = \frac{1}{j^2}$$

and thus $\mathsf{P}\left(\liminf_j C_j\right) = 1$. Now,

$$\begin{aligned}
\mathsf{P}\left(\overline{D_j}\right) &\leq \frac{1}{j^2} + \mathsf{P}(\overline{D_j} \cap C_j) \\
&= \frac{1}{j^2} + \mathsf{P}\left(\cup_{m\geq 0}(\overline{D_j^m} \cap C_j)\right) \\
&\leq \frac{1}{j^2} + \sum_{m=0}^{+\infty} \mathsf{P}\left(\overline{D_j^m} \cap C_j\right).
\end{aligned}$$

We consider several cases, depending on the respective values of j and m :

- If $m > j + \frac{2}{\log(2)}\log j$,

$$\mathsf{P}\left(\overline{D_j^m} \cap C_j\right) = 0.$$

- If $j + \frac{2}{\log(2)}\log j \geq m \geq j$,

$$\mathsf{P}\left(\overline{D_j^m}\right) \leq \mathsf{P}\left(\sup_{\frac{1}{2^{m+1}} \leq |v-u| \leq \frac{1}{2^m}} \left|\sum_{i=2^j}^{2^{j+1}-1} \gamma_i i^{-1/\alpha(u)} \mathbf{1}_{[u,v]}(V_i)\right| \geq \frac{1}{2^{(m+1)\eta} j^2}\right).$$

Let $J_0 \in \mathbb{N}$ be such that for all $j > J_0$, $2^{j(\frac{1}{\alpha(u)}-\eta)} > 2^{\eta} j^{3+\frac{2\eta}{\log(2)}}$. The event :

$$\{\sup_{\frac{1}{2^{m+1}} \leq |v-u| \leq \frac{1}{2^m}} \left|\sum_{i=2^j}^{2^{j+1}-1} \gamma_i i^{-1/\alpha(u)} \mathbf{1}_{[u,v]}(V_i)\right| \geq \frac{1}{2^{(m+1)\eta} j^2}\}$$

is included in the event

$$\cup_{N\geq 1}^{2^j}\left(\cup_{l_1,\ldots,l_N \in [\![2^j, 2^{j+1}-1]\!]}\{|\sum_{i=1}^{N} \gamma_{l_i} l_i^{-1/\alpha(u)}| > \frac{1}{2^{(m+1)\eta} j^2}\} \cap (\cap_{i=1}^{N}\{|V_{l_i} - u| \in [\frac{1}{2^{m+1}}, \frac{1}{2^m}]\}) \ldots\right.$$

$$\left. \ldots \cap (\cap_{k\neq l_i}\{|V_k - u| \notin [\frac{1}{2^{m+1}}, \frac{1}{2^m}]\})\right).$$

Notice that for $j \geq J_0$ and $N < j$, $\mathsf{P}\left(|\sum_{i=1}^{N}\gamma_{l_i} l_i^{-1/\alpha(u)}| > \frac{1}{2^{(m+1)\eta}j^2}\right) = 0$, and thus

$$
\begin{aligned}
\mathsf{P}\left(\overline{D_j^m}\right) &\leq \sum_{N=j}^{2^j} \sum_{l_1,\ldots,l_N \in [\![2^j,2^{j+1}-1]\!]} \mathsf{P}\left(|\sum_{i=1}^{N}\gamma_{l_i} l_i^{-1/\alpha(u)}| > \frac{1}{2^{(m+1)\eta}j^2}\right) \mathsf{P}\left(\cap_{i=1}^{N}\{|V_{l_i}-u| \in [\frac{1}{2^{m+1}},\frac{1}{2^m}]\}\right) \\
&\leq \sum_{N=j}^{2^j} \frac{1}{2^{(m+1)N}} \sum_{l_1,\ldots,l_N \in [\![2^j,2^{j+1}-1]\!]} \mathsf{P}\left(|\sum_{i=1}^{N}\gamma_{l_i} l_i^{-1/\alpha(u)}| > \frac{1}{2^{(m+1)\eta}j^2}\right) \\
&\leq \sum_{N=j}^{2^j} \frac{1}{2^{(m+1)N}} \sum_{l_1,\ldots,l_N \in [\![2^j,2^{j+1}-1]\!]} j^4 2^{2(m+1)\eta} \sum_{i=2^j}^{2^{j+1}-1} \frac{1}{i^{\frac{2}{\alpha(u)}}} \\
&\leq \sum_{N=j}^{2^j} \frac{j^4 2^{2(m+1)\eta}}{2^{(m+1)N}} 2^{j(1-\frac{2}{\alpha(u)})} C_{2^j}^N \\
&\leq j^4 2^{2(j+\frac{2}{\log(2)}\log j+1)\eta - j\frac{2}{\alpha(u)}} \sum_{N=j}^{2^j} \frac{2^j C_{2^j}^N}{2^{(m+1)N}} \\
&\leq j^{4+\frac{4\eta}{\log(2)}} 2^{2j(\eta-\frac{1}{\alpha(u)})} \sum_{N=j}^{2^j} \frac{2^{j-N}2^{(j-m)N}}{N!} \\
&\leq 3j^{4+\frac{4\eta}{\log(2)}} 2^{2j(\eta-\frac{1}{\alpha(u)})}.
\end{aligned}
$$

- When $j \geq m \geq \frac{\log(j)}{\log(2)}$, the same computations lead to :

$$
\begin{aligned}
&\sum_{N=j2^{j-m}}^{2^j} \sum_{l_1,\ldots,l_N \in [\![2^j,2^{j+1}-1]\!]} \mathsf{P}\left(|\sum_{i=1}^{N}\gamma_{l_i} l_i^{-1/\alpha(u)}| > \frac{1}{2^{(m+1)\eta}j^2}\right) \mathsf{P}\left(\cap_{i=1}^{N}\{|V_{l_i}-u| \in [\frac{1}{2^{m+1}},\frac{1}{2^m}]\}\right) \\
\leq\ & \sum_{N=j2^{j-m}}^{2^j} \frac{j^4 2^{2(m+1)\eta}}{2^{(m+1)N}} 2^{j(1-\frac{2}{\alpha(u)})} C_{2^j}^N \\
\leq\ & j^4 2^{2(m+1)\eta - 2j/\alpha(u)} \sum_{N=j2^{j-m}}^{2^j} \frac{2^{j-N}2^{(j-m)N}}{N!} \\
\leq\ & j^4 2^{2\eta} 2^{2j(\eta-\frac{1}{\alpha(u)})} \sum_{N=j2^{j-m}}^{+\infty} \frac{2^{(j-m)N}}{N!} \\
\leq\ & K j^4 2^{2j(\eta-\frac{1}{\alpha(u)})} \frac{e^{2^{j-m}} 2^{(j-m)(j2^{j-m}+1)}}{(j2^{j-m}+1)!}
\end{aligned}
$$

where we have used the estimate $\sum_{n\geq N}\frac{x^n}{n!}\leq e^x\frac{x^{N+1}}{(N+1)!}$. We arrive at :

$$\begin{aligned}\mathsf{P}\left(\overline{D_j^m}\right) &\leq Kj^4 2^{2j(\eta-\frac{1}{\alpha(u)})}+\\ &\sum_{N=1}^{j2^{j-m}}\frac{1}{2^{(m+1)N}}(1-\frac{1}{2^{m+1}})^{2^j-N}\sum_{l_1,\dots,l_N\in[\![2^j,2^{j+1}-1]\!]}\mathsf{P}\left(|\sum_{i=1}^N\gamma_{l_i}l_i^{-1/\alpha(u)}|>\frac{1}{2^{(m+1)\eta}j^2}\right).\end{aligned}$$

We need to distinguish two cases depending on the value of η. If $\eta\leq\frac{1}{2}$, fix $J_1\in\mathbb{N}$ such that for all $j\geq J_1$, $2^{j(\frac{1}{\alpha(u)}-\frac{1}{2})}>2^{1/\alpha(u)}j^3\sqrt{j}$. If $\eta>\frac{1}{2}$, fix $J_1\in\mathbb{N}$ such that for all $j\geq J_1$, $2^{j(\frac{1}{\alpha(u)}-\eta)}>2^{1/\alpha(u)}j^3\sqrt{j}$. Then for all η and all $j\geq J_1$, one has $\frac{2^{j/\alpha(u)}}{j^3\sqrt{j2^{j-m}}2^{(m+1)\eta}}\geq 1$ and

$$\begin{aligned}\mathsf{P}\left(|\sum_{i=1}^N\gamma_{l_i}l_i^{-1/\alpha(u)}|>\frac{1}{2^{(m+1)\eta}j^2}\right) &\leq \mathsf{P}\left(\left|\sum_{i=1}^N\gamma_{l_i}\left(\frac{2^j}{l_i}\right)^{1/\alpha(u)}\right|>j\sqrt{N}\right)\\ &\leq 2e^{-j^2/2}.\end{aligned}$$

We then get

$$\begin{aligned}\mathsf{P}\left(\overline{D_j^m}\right) &\leq Kj^4 2^{2j(\eta-\frac{1}{\alpha(u)})}+\sum_{N=1}^{j2^{j-m}}\frac{1}{2^{(m+1)N}}(1-\frac{1}{2^{m+1}})^{2^j-N}C_{2^j}^N 2e^{-j^2/2}\\ &\leq Kj^4 2^{2j(\eta-\frac{1}{\alpha(u)})}+2e^{-j^2/2}\sum_{N=1}^{2^j}\frac{1}{2^{(m+1)N}}(1-\frac{1}{2^{m+1}})^{2^j-N}C_{2^j}^N\\ &\leq Kj^4 2^{2j(\eta-\frac{1}{\alpha(u)})}+2e^{-j^2/2}.\end{aligned}$$

- Assume finally that $m\leq\frac{\log(j)}{\log(2)}$.

Fix $J_2\in\mathbb{N}$ such that for all $j\geq J_2$, $2^{j(\frac{1}{\alpha(u)}-\frac{1}{2})}>2^{1/\alpha(u)}j^{3+\eta}$. Then, for $j\geq J_2$, one has $\frac{2^{j/\alpha(u)}}{j^3\sqrt{2^j}2^{(m+1)\eta}}\geq 1$ and computations similar the ones above lead to

$$\begin{aligned}\mathsf{P}\left(\overline{D_j^m}\right) &\leq \sum_{N=1}^{2^j}\frac{1}{2^{(m+1)N}}(1-\frac{1}{2^{m+1}})^{2^j-N}\sum_{l_1,\dots,l_N\in[\![2^j,2^{j+1}-1]\!]}\mathsf{P}\left(|\sum_{i=1}^N\gamma_{l_i}l_i^{-1/\alpha(u)}|>\frac{1}{2^{(m+1)\eta}j^2}\right)\\ &\leq \sum_{N=1}^{2^j}\frac{1}{2^{(m+1)N}}(1-\frac{1}{2^{m+1}})^{2^j-N}\sum_{l_1,\dots,l_N\in[\![2^j,2^{j+1}-1]\!]}\mathsf{P}\left(|\sum_{i=1}^N\gamma_{l_i}(\frac{2^{j/\alpha(u)}}{l_i})^{1/\alpha(u)}|>j\sqrt{N}\right)\\ &\leq 2e^{-j^2/2}.\end{aligned}$$

We thus get that, for $j \geq \max(J_0, J_1, J_2)$,

$$\sum_{m=0}^{+\infty} \mathsf{P}\left(\overline{D_j^m} \cap C_j\right) \quad \leq K \log(j) j^{4+\frac{4\eta}{\log(2)}} 2^{2j(\eta - \frac{1}{\alpha(u)})},$$

and thus $\mathsf{P}(\liminf_j D_j) = 1$.

On the event $\liminf_j C_j \cap \liminf_j D_j$, we may fix $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,

$$\sup_{v \in [0,1]} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i i^{-1/\alpha(u)} \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta} \right| \leq \frac{1}{j^2}.$$

Since $\sup_{v \in [0,1]} \left| \sum_{i=1}^{2^{j_0}-1} \gamma_i i^{-1/\alpha(u)} \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta} \right| < +\infty$, we obtain

$$\sup_{v \in [0,1]} \left| \sum_{i=1}^{+\infty} \gamma_i i^{-1/\alpha(u)} \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta} \right| < +\infty.$$

Let us now deal with

$$E_j = \left\{ \sup_{v \in [0,1]} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i (\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}) \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta} \right| \leq \frac{1}{j^2} \right\}.$$

$$\begin{aligned}
\mathsf{P}\left(\overline{E_j}\right) &\leq \frac{1}{j^2} + \mathsf{P}(\overline{E_j} \cap C_j) \\
&\leq \frac{1}{j^2} + \mathsf{P}\left(2^{j\eta} j^{2\eta} \sup_{v \in [0,1]} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i (\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}) \mathbf{1}_{[u,v]}(V_i) \right| > \frac{1}{j^2} \right) \\
&\leq \frac{1}{j^2} + \mathsf{P}\left(\sum_{i=2^j}^{2^{j+1}-1} \left| (\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}) \right| > \frac{1}{2^{j\eta} j^{2(1+\eta)}} \right) \\
&\leq \frac{1}{j^2} + 2^{j\eta} j^{2(1+\eta)} \sum_{i=2^j}^{2^{j+1}-1} \mathsf{E}|\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}| \\
&\leq \frac{1}{j^2} + 2^{j\eta} j^{2(1+\eta)} \sum_{i=2^j}^{2^{j+1}-1} 2(\mathsf{P}(\Gamma_i < \frac{i}{2}) + \mathsf{P}(\Gamma_i > 2i)) \\
&\quad + 2^{j\eta} j^{2(1+\eta)} \sum_{i=2^j}^{2^{j+1}-1} \mathsf{E}|\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}| \mathbf{1}_{\{\frac{i}{2} < \Gamma_i < 2i\}}.
\end{aligned}$$

However

$$\begin{aligned}\mathsf{E}|\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}|\mathbf{1}_{\{\frac{i}{2}<\Gamma_i<2i\}} &\leq \frac{1}{i^{1/\alpha(u)}} K_u \mathsf{E}|\frac{\Gamma_i}{i} - 1| \\ &\leq K_u \frac{1}{i^{1+\frac{1}{\alpha(u)}}}\end{aligned}$$

and

$$2^{j\eta} j^{2(1+\eta)} \sum_{i=2^j}^{2^{j+1}-1} \mathsf{E}|\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}|\mathbf{1}_{\{\frac{i}{2}<\Gamma_i<2i\}} \leq K j^{2(1+\eta)} 2^{j(\eta - \frac{1}{\alpha(u)})}.$$

We thus obtain $\mathsf{P}\left(\liminf_j E_j\right) = 1$. As a consequence, $\sup_{v\in[0,1]} \left|\sum_{i=1}^{+\infty} \gamma_i (\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}) \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta}\right| < +\infty$ and finally

$$\sup_{v\in[0,1]} \left| \frac{X(v,u) - X(u,u)}{|v-u|^\eta} \right| < +\infty \quad \blacksquare$$

Lemma IV.18. For all $u \in (0,1)$, one has almost surely, for all $\eta \in (0, \frac{1}{\alpha(u)})$,

$$\sup_{v\in[0,1]} \frac{|X(v,u) - X(u,u)|}{|v-u|^\eta} < +\infty.$$

Proof

Fix $u \in (0,1)$. Lemma IV.17 yields that, for all $\eta \in (0, \frac{1}{\alpha(u)})$, we may choose an Ω_η having probability one and such that, on Ω_η, $\sup_{v\in[0,1]} \left|\frac{X(v,u)-X(u,u)}{|v-u|^\eta}\right| < +\infty$. Thus, on $\Omega = \cap_{j\geq 0} \Omega_{\frac{1}{\alpha(u)} - \frac{1}{2^j}}$, which still has probability one, it holds that, for all $\eta \in (0, \frac{1}{\alpha(u)})$, $\sup_{v\in[0,1]} \frac{|X(v,u)-X(u,u)|}{|v-u|^\eta} < +\infty$ $\blacksquare$

Proof of Theorem IV.7

From Theorem IV.6, we already know that $\mathcal{H}_u \leq \frac{1}{\alpha(u)}$. To prove the reverse inequality, we treat separately the situations where $\alpha < 1$ and $\alpha \geq 1$.

• Consider first the case $0 < \alpha(u) < 1$.

Write :

$$Y(v) - Y(u) = X(v,v) - X(v,u) + X(v,u) - X(u,u).$$

By Lemma IV.18, we know that the Hölder regularity of $v \mapsto X(v,u) - X(u,u)$ at u is almost surely not smaller than $\frac{1}{\alpha(u)}$. Now, by applying the finite increments theorem to the functions $t \mapsto C_t^{1/t} \Gamma_i^{-1/t}$, we get

$$\begin{aligned} X(v,v) - X(v,u) &= \sum_{i=1}^{\infty} \gamma_i \mathbf{1}_{[0,v]}(V_i) \left(C_{\alpha(v)}^{1/\alpha(v)} \Gamma_i^{-1/\alpha(v)} - C_{\alpha(u)}^{1/\alpha(u)} \Gamma_i^{-1/\alpha(u)} \right) \\ &= (\alpha(v) - \alpha(u)) \sum_{i=1}^{\infty} \gamma_i \mathbf{1}_{[0,v]}(V_i) \left(CP(\alpha(w_i)) - C_{\alpha(w_i)}^{1/\alpha(w_i)} \frac{\log \Gamma_i}{\alpha(w_i)^2} \right) \Gamma_i^{-1/\alpha(w_i)}, \end{aligned}$$

where, for each i, $w_i \in [u,v]$ (or $[v,u]$), and CP denotes the derivative of the function $t \mapsto C_t^{1/t}$. However,

$$\begin{aligned} |\sum_{i=1}^{\infty} \gamma_i \mathbf{1}_{[0,v]}(V_i) \left(Cp(\alpha(w_i)) - \frac{\log \Gamma_i}{\alpha(w_i)^2} \right) \Gamma_i^{-1/\alpha(w_i)}| &\leq \sum_{i=1}^{\infty} \left| CP(\alpha(w_i)) - C_{\alpha(w_i)}^{1/\alpha(w_i)} \frac{\log \Gamma_i}{\alpha(w_i)^2} \right| \Gamma_i^{-1/\alpha(w_i)} \\ &\leq K \sum_{i=1}^{\infty} (1 + |\log \Gamma_i|) \left(\Gamma_i^{-1/c} + \Gamma_i^{-1/d} \right). \end{aligned}$$

Thus the quantity $T(u,v) = \sum_{i=1}^{\infty} \gamma_i \mathbf{1}_{[0,v]}(V_i) \left(CP(\alpha(w_i)) - C_{\alpha(w_i)}^{1/\alpha(w_i)} \frac{\log \Gamma_i}{\alpha(w_i)^2} \right) \Gamma_i^{-1/\alpha(w_i)}$ is, uniformly in v, almost surely finite and not 0. As a consequence, the function $v \mapsto X(v,v) - X(v,u) = (\alpha(u) - \alpha(v))T(u,v)$ has almost surely the same Hölder exponent at u as the function $v \mapsto \alpha(v)$ at u. If $\mathcal{H}_u^\alpha < \frac{1}{\alpha(u)}$, this entails that Y has exponent $\mathcal{H}_u^\alpha$ at u. If $\mathcal{H}_u^\alpha > \frac{1}{\alpha(u)}$, then the exponent of Y at u is at least $\frac{1}{\alpha(u)}$ and thus exactly $\frac{1}{\alpha(u)}$ by Theorem IV.6.

- Assume now that $1 \leq \alpha(u) < 2$.

Let $\eta < \frac{1}{\alpha(u)}$ and $\delta \in (\eta, \frac{1}{\alpha(u)})$. Then :

$$\frac{|Y(v) - Y(u)|}{|v-u|^\eta} \leq \frac{|X(v,v) - X(v,u)|}{|v-u|^\eta} + \frac{|X(v,u) - X(u,u)|}{|v-u|^\eta}.$$

By Lemma IV.18, there exists $K > 0$ such that $\frac{|X(v,u)-X(u,u)|}{|v-u|^\eta} \leq K|v-u|^{\delta-\eta}$, and, by Lemma IV.16, there exists $K > 0$ such that $\frac{|X(v,v)-X(v,u)|}{|v-u|^\eta} \leq K|v-u|^{1-\eta}$. This entails $\lim_{v \to u} \frac{|Y(v)-Y(u)|}{|v-u|^\eta} = 0$ and

$$\mathcal{H}_u \geq \frac{1}{\alpha(u)} \qquad \blacksquare$$

IV.5 Assumptions

This section gathers the various conditions required on the considered processes so that our results hold. For all the assumptions, we shall denote $c = \inf_{v \in U} \alpha(v)$ and $d = \sup_{v \in U} \alpha(v)$.

- (C1) The family of functions $v \to f(t,v,x)$ is differentiable for all (v,t) in U^2 and almost all x in E. The derivatives of f with respect to v are denoted by f'_v.
- (C2) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbb{R}} \left[\sup_{w \in U} (|f(t,w,x)|^{\alpha(w)}) \right]^{1+\delta} \hat{m}(dx) < \infty.$$

- (Cs2) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbb{R}} \left[\sup_{w \in U} (|f(t,w,x)|^{\alpha(w)}) \right]^{1+\delta} r(x)^{\delta} \, m(dx) < \infty.$$

- (C3) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbb{R}} \left[\sup_{w \in U} (|f'_v(t,w,x)|^{\alpha(w)}) \right]^{1+\delta} \hat{m}(dx) < \infty.$$

- (Cs3) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbb{R}} \left[\sup_{w \in U} (|f'_v(t,w,x)|^{\alpha(w)}) \right]^{1+\delta} r(x)^{\delta} \, m(dx) < \infty.$$

- (Cs4) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbb{R}} \left[\sup_{w \in U} \left[|f(t,w,x) \log(r(x))|^{\alpha(w)} \right] \right]^{1+\delta} r(x)^{\delta} \, m(dx) < \infty.$$

- (C5) $X(t,u)$ (as a process in t) is localisable at u with exponent $h(u) \in (h_-, h_+) \subset (0,1)$, with local form $X'_u(t,u)$, and $u \mapsto h(u)$ is a C^1 function .
- (C6) There exists $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbb{R}$,

$$|f(v,u,x)| \leq K_U.$$

- (C7) There exists $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbb{R}$,

$$\left| f'_v(v,u,x) \right| \leq K_U.$$

- (C8) There exists a function h defined on U, $\varepsilon_0 \in (0,1)$ and $K_U > 0$ such that $\forall r < \varepsilon_0$, $\forall x \in \mathbb{R}$,

$$\frac{1}{r^{h(t)-1/\alpha(t)}} |f(t+r,t,x) - f(t,t,x)| \leq K_U.$$

- (Cu8) There exists a function h defined on U and $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbb{R}$,

$$\frac{1}{|v-u|^{h(u)-1/\alpha(u)}} |f(v,u,x) - f(u,u,x)| \leq K_U.$$

- (C9) There exists a function h defined on U, $\varepsilon_0 > 0$ and $K_U > 0$ such that $\forall r < \varepsilon_0$,

$$\frac{1}{r^{h(t)\alpha(t)}} \int_{\mathbb{R}} |f(t+r,t,x) - f(t,t,x)|^{\alpha(t)} m(dx) \leq K_U.$$

- (C10) There exists a function h defined on U and $p \in (\alpha(t), 2)$, $p \geq 1$, such that for all $\varepsilon > 0$, there exists $K_U > 0$ such that, $\forall r \leq \varepsilon$,

$$\frac{1}{r^{1+p(h(t)-\frac{1}{\alpha(t)})}} \int_{\mathbb{R}} |f(t+r,t,x) - f(t,t,x)|^{p} \, m(dx) \leq K_U.$$

- (Cu10) There exists a function h defined on U, $p \in (d,2)$, $p \geq 1$ and $K_U > 0$ such that $\forall v \in U, \forall u \in U$,

$$\frac{1}{|v-u|^{1+p(h(u)-\frac{1}{\alpha(u)})}} \int_{\mathbb{R}} |f(v,u,x) - f(u,u,x)|^p \, m(dx) \leq K_U.$$

- (C11) $\forall \varepsilon > 0$, $\exists K_U > 0$ such that, $\forall r \leq \varepsilon$,

$$\int_{\mathbb{R}} |f(t+r,t,x)|^2 \, m(dx) \leq K_U.$$

- (Cu11) There exists $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$,

$$\int_{\mathbb{R}} |f(v,u,x)|^2 \, m(dx) \leq K_U.$$

- (C12) $\forall \varepsilon > 0$, $\exists K_U > 0$ such that $\forall r \leq \varepsilon$,

$$\int_{\mathbb{R}} |f(t+r,t+r,x)|^2 \, m(dx) \leq K_U.$$

- (Cu12) There exists $K_U > 0$ such that $\forall v \in U$,

$$\int_{\mathbb{R}} |f(v,v,x)|^2 \, m(dx) \leq K_U.$$

- (C13)

$$\inf_{v \in U} \int_{\mathbb{R}} f(v,v,x)^2 m(dx) > 0.$$

- (C14) There exists a function h and a positive function g defined on U such that

$$\lim_{r \to 0} \frac{1}{r^{1+2(h(t)-1/\alpha(t))}} \int_{\mathbb{R}} \left(f(t+r,t,x) - f(t,t,x)\right)^2 m(dx) = g(t).$$

- (Cu14) There exists a function h and a positive function g defined on U such that

$$\lim_{r \to 0} \sup_{t \in U} \left| \frac{1}{r^{1+2(h(t)-1/\alpha(t))}} \int_{\mathbb{R}} \left(f(t+r,t,x) - f(t,t,x)\right)^2 m(dx) - g(t) \right| = 0.$$

- (C15) $\forall \varepsilon > 0$, $\exists K_U > 0$ such that $\forall r \leq \varepsilon$,

$$\frac{1}{|r|^2} \int_{\mathbb{R}} |f(t+r,t+r,x) - f(t+r,t,x)|^2 \, m(dx) \leq K_U.$$

- (Cu15) $\exists K_U > 0$ such that, $\forall v \in U$, $\forall u \in U$,

$$\frac{1}{|v-u|^2} \int_{\mathbb{R}} |f(v,v,x) - f(v,u,x)|^2 \, m(dx) \leq K_U.$$

Chapitre V

Estimation de la fonction de stabilité et de la fonction de localisabilité des processus multistables

Plan du chapitre

Abstract

In this work, we give two estimators of the stability and the localisability functions, and we prove the consistency of those two estimators. We illustrate these convergences with two classical examples, the Levy multistable process and the Linear Multifractional Multistable Motion.

V.1 Construction of the estimators

Let Y be a multistable process defined in (I.21). The estimation of the localisability function and the stability function is based on the increments $(Y_{k,N})$ of Y. Define the sequence $(Y_{k,N})_{k\in\mathbb{Z},N\in\mathbb{N}}$ by

$$Y_{k,N} = Y(\frac{k+1}{N}) - Y(\frac{k}{N}).$$

Let $t_0 \in \mathbb{R}$ fixed. We introduce an estimator of $H(t_0)$ with

$$\hat{H}_N(t_0) = -\frac{1}{n(N)\log N}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}\log|Y_{k,N}|$$

where $(n(N))_{N\in\mathbb{N}}$ is a sequence taking even integer values. We expect the sequence $(\hat{H}_N(t_0))_N$ to converge to $H(t_0)$ thanks to the localisability of the process Y. For the integers k and N such that $\frac{k}{N}$ is close to t_0, $\frac{Y_{k,N}}{(\frac{1}{N})^{H(t_0)}}$ is asymptotically distributed as $Y'_{t_0}(1)$. We have then $-\frac{\log|Y_{k,N}|}{\log N} = H(t_0)+\frac{Z_{k,N}}{\log N}$ where $(Z_{k,N})_{k,N}$ converge weakly to $-\log|Y'_{t_0}(1)|$ when N tends to infinity and $\dfrac{k}{N}$ tends to t_0. We regulate the sequence $(Z_{k,N})$ near t_0 using the mean $\frac{1}{n(N)}\sum\limits_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} Z_{k,N}$ and we can expect this sum will be bounded in the L^r spaces to obtain the convergence with a rate $\frac{1}{\log N}$. The convergence is proved in Theorem V.1.

Let $p_0 \in (0,2)$ and $\gamma \in (0,1)$. With the increments of the process, we define the empirical moments $S_N(p)$ by

$$S_N(p) = \left(\frac{1}{n(N)}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}|Y_{k,N}|^p\right)^{\frac{1}{p}}.$$

Let

$$R_{\exp}(p) = \frac{S_N(p_0)}{S_N(p)} \text{ and } R_\alpha(p) = \frac{(\mathsf{E}|Z|^{p_0})^{1/p_0}}{(\mathsf{E}|Z|^p)^{1/p}}\mathbf{1}_{p<\alpha}$$

where Z is a standard symmetric α-stable random variable (written $Z \sim S_\alpha(1,0,0)$ as in [49]), i.e $\mathsf{E}|Z|^p = \frac{2^{p-1}\Gamma(1-\frac{p}{\alpha})}{p\int_0^{+\infty}u^{-p-1}\sin^2(u)du}$.

Consider the set $A_N =: \underset{\alpha\in[0,2]}{\arg\min}\left(\int_{p_0}^2|R_{\exp}(p)-R_\alpha(p)|^\gamma dp\right)^{1/\gamma}$. Since the function $\alpha \to \left(\int\limits_{p_0}^{2}|R_{\exp}(p)-R_\alpha(p)|^\gamma dp\right)^{1/\gamma}$ is a continuous function, A_N is a non empty closed set. We define then an estimator of $\alpha(t_0)$ by

$$\hat{\alpha}_N(t_0) = \min\left(\arg\min_{\alpha\in[0,2]}\left(\left(\int_{p_0}^{2}|R_{\exp}(p)-R_\alpha(p)|^\gamma dp\right)^{1/\gamma}\right)\right).$$

Under the conditions of Theorem V.2, Y is $H(t_0)$-localisable and $Y'_{t_0}(1) \sim S_{\alpha(t_0)}(1,0,0)$ so $\frac{|Y_{k,N}|^p}{(\frac{1}{N})^{pH(t_0)}}$ converge weakly to $|Y'_{t_0}(1)|^p$ and with a meaning effect, $N^{H(t_0)}S_N(p)$ tends to $(\mathsf{E}|Y'_{t_0}(1)|^p)^{1/p}$ in probability, which is the result of Theorem V.2. Without more conditions, $\int_{p_0}^{2}|R_{\exp}(p)-R_\alpha(p)|^\gamma dp$ tends to $\int_{p_0}^{2}|R_{\alpha(t_0)}(p)-R_\alpha(p)|^\gamma dp$. Naturally, $\alpha(t_0)$ is the only solution of $\arg\min_{\alpha\in[0,2]}\int_{p_0}^{2}|R_{\alpha(t_0)}(p)-R_\alpha(p)|^\gamma dp$ and this leads to the definition of $\hat{\alpha}_N(t_0)$. The convergence is proved in Theorem V.3.

V.2 Main results

The three following theorems apply to a diagonal process Y defined from the field X given by (I.20). For convenience, the conditions required on X and the function f that appears in (I.20), denoted (C1), ..., (C14), are gathered in Section V.5. Theorem V.1 lead to the convergence in the L^r spaces of the estimator of the localisability function H, while the two Theorems V.2 and V.3 draw to the convergence of the estimator of the stability function α.

V.2.1 Approximation of the localisability function

Theorem V.1. *Let Y a multistable process. Assume the conditions (C1), (C2), (C3) (or (C1), (Cs2), (Cs3) and (Cs4) in the σ-finite space case), and that there exists a function H such that (C5)-(C14) hold. Assume in addition that $\lim\limits_{N\to+\infty}\frac{N}{n(N)}=+\infty$.*

Then, for all $t_0 \in U$ and all $r > 0$,

$$\lim_{N\to+\infty}\mathsf{E}\left|\hat{H}_N(t_0)-H(t_0)\right|^r = 0.$$

Proof
See Section V.4.

Remark : Under the conditions (C1), (C2), (C3) and (C5) listed in the theorem, Theorems 3.3 and 4.5 of [29] imply that Y is $H(t_0)-$localisable at t_0.

V.2.2 Approximation of the stability function

We first give conditions for the convergence in probability of $S_N(p)$ in Theorem V.2, which is useful to establish the consistency of the estimator $\hat{\alpha}_N(t_0)$.

Theorem V.2. *Let Y a multistable process. Assume the conditions (C1), (C2), (C3) (or (C1), (Cs2), (Cs3) and (Cs4) in the σ-finite space case). Assume in addition that :*

- $\lim\limits_{N\to+\infty} n(N) = +\infty$.

- $\lim\limits_{N\to+\infty} \frac{N}{n(N)} = +\infty$.
- *The process $X(.,t_0)$ is $H(t_0)$-self-similar with stationary increments and $H(t_0) < 1$.*
- *(C*) There exists $\epsilon_1 > 0$ and $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,*

$$\int_E |h_{0,t_0}(x)h_{j,t_0}(x)l|^{\frac{\alpha(t_0)}{2}} m(dx) \leq (1-\epsilon_1)\|h_{0,t_0}\|_{\alpha(t_0)}^{\alpha(t_0)},$$

  where $h_{j,u}(x) = f(j+1,u,x) - f(j,u,x)$.
- $\lim\limits_{j\to+\infty} \int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} m(dx) = 0.$

Then, for all $p \in [p_0, \alpha(t_0))$,

$$N^{H(t_0)} S_N(p) \underset{N\to+\infty}{\longrightarrow} (\mathsf{E}|Z|^p)^{1/p}$$

where the convergence is in probability and $Z \sim S_{\alpha(t_0)}(1,0,0)$.

Proof
See Section V.4.

Theorem V.3. *Let Y a multistable process. Assume the conditions of Theorem V.2, then, for all $t_0 \in U$ and $r > 0$,*

$$\lim_{N\to+\infty} \mathsf{E}\,|\hat{\alpha}_N(t_0) - \alpha(t_0)|^r = 0.$$

Proof
See Section V.4.

V.3 Examples and simulations

In this section, we consider the “multistable versions” of some classical processes : the α-stable Lévy motion and the Linear Fractional Stable Motion.

We first recall some definitions. In the sequel, M will denote a symmetric α-stable ($0 < \alpha < 2$) random measure on $\mathbb{R}$ with control measure Lebesgue measure $\mathcal{L}$. We will write

$$L_\alpha(t) := \int_0^t M(dz)$$

for α-stable Lévy motion, and we will use the Ferguson-Klass-LePage representation,

$$\forall t \in (0,1),\ \ L_\alpha(t) = C_\alpha^{1/\alpha} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha} \mathbf{1}_{[0,t]}(V_i).$$

The following process is called *linear fractional α-stable motion* :

$$L_{\alpha,H,b^+,b^-}(t) = \int_{-\infty}^{\infty} f_{\alpha,H}(b^+,b^-,t,x) M(dx)$$

where $t \in \mathbb{R}$, $H \in (0,1)$, $b^+, b^- \in \mathbb{R}$, and

$$f_{\alpha,H}(b^+, b^-, t, x) = b^+ \left((t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}\right) \\ + b^- \left((t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha}\right).$$

When $b^+ = b^- = 1$, this process is called well-balanced linear fractional α-stable motion and denoted $L_{\alpha,H}$.

The localisability of Lévy motion and linear fractional α-stable motion simply stems from the fact that they are $1/\alpha$-self-similar with stationary increments [14].

We now apply our results to the multistable versions of these processes, that were defined in [15, 16].

V.3.1 Symmetric multistable Lévy motion

Let $\alpha : [0,1] \to [c,d] \subset (1,2)$ be continuously differentiable. Define

$$X(t,u) = C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} \mathbf{1}_{[0,t]}(V_i) \tag{V.1}$$

and the symmetric multistable Lévy motion

$$Y(t) = X(t,t).$$

Proposition V.4. If $\lim_{N\to+\infty} n(N) = +\infty$ and $\lim_{N\to+\infty} \frac{N}{n(N)} = +\infty$, then for all $r > 0$,

$$\lim_{N\to+\infty} \mathsf{E} \left| \hat{H}_N(t_0) - \frac{1}{\alpha(t_0)} \right|^r = 0 \text{ and } \lim_{N\to+\infty} \mathsf{E} \left| \hat{\alpha}_N(t_0) - \alpha(t_0) \right|^r = 0.$$

Proof

We know from [30] that all the conditions (C1)-(C14) are satisfied. We deduce from Theorem V.1 that $\lim_{N\to+\infty} \mathsf{E} \left| \hat{H}_N(t_0) - \frac{1}{\alpha(t_0)} \right|^r = 0$. Since the process $X(.,t_0)$ is a Lévy motion $\alpha(t_0)$-stable, $X(.,t_0)$ is $\frac{1}{\alpha(t_0)}$-self-similar with stationary increments [49]. We then prove that the condition (C*) is satisfied.

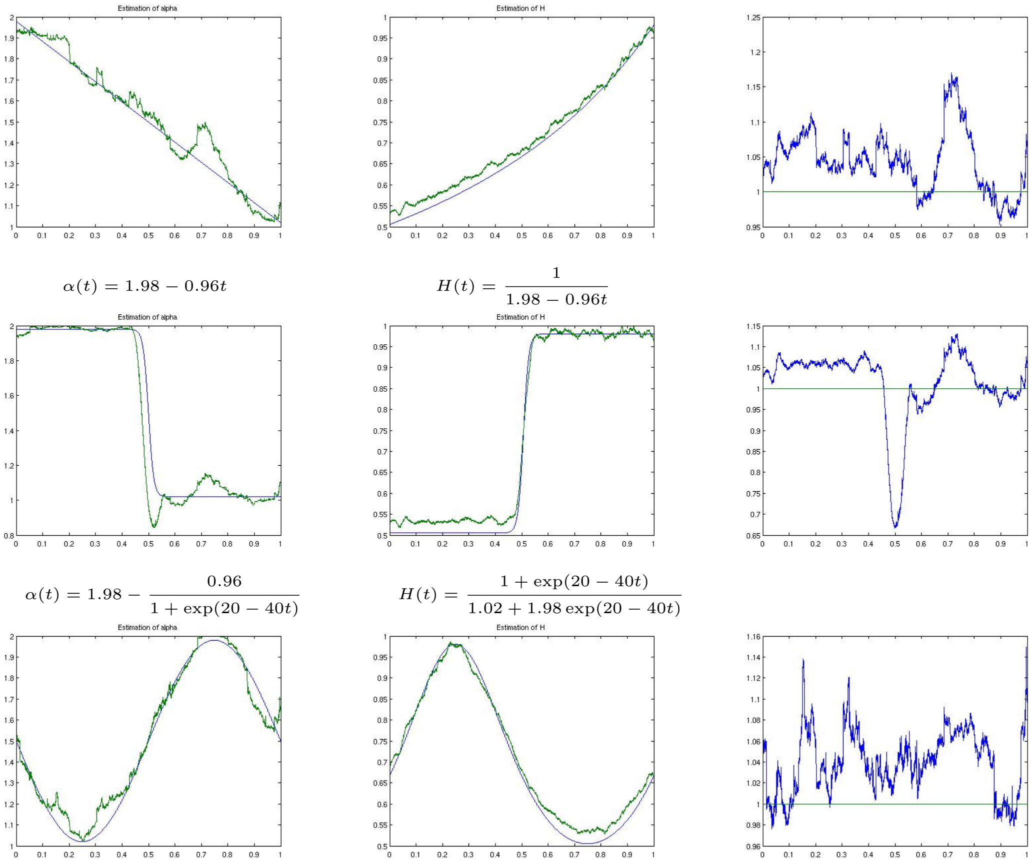
$$h_{j,t_0}(x) = \mathbf{1}_{[j,j+1[}(x)$$

so for $j \geq 1$,

$$\int_{\mathbb{R}} |h_{0,t_0}(x) h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} dx = 0.$$

We conclude with Theorem V.3 ■

We display on Figure V.1 some examples of estimations for various functions α, the function H satisfying the relation $H(t) = \frac{1}{\alpha(t)}$. The trajectories have been simulated using the field (V.1). For each $u \in (0,1)$, $X(.,u)$ is a $\alpha(u)$-stable Lévy Motion. It is then an $\alpha(u)$-stable process with independent increments. We have generated these increments using the RSTAB program available in [53] or in [49], and then taken the diagonal $X(t,t)$.



$\alpha(t) = 1.98 - 0.96t$ $\qquad H(t) = \dfrac{1}{1.98 - 0.96t}$

$\alpha(t) = 1.98 - \dfrac{0.96}{1 + \exp(20 - 40t)}$ $\qquad H(t) = \dfrac{1 + \exp(20 - 40t)}{1.02 + 1.98\exp(20 - 40t)}$

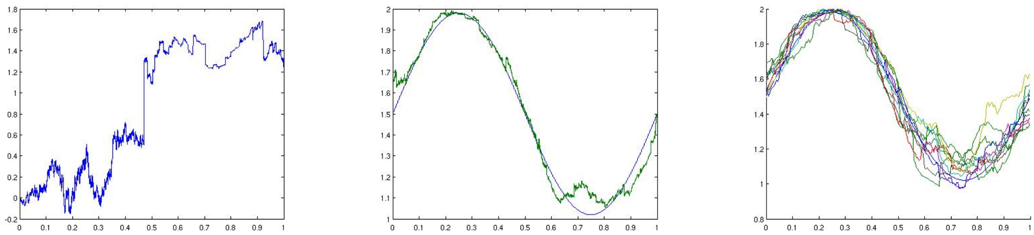
$\alpha(t) = 1.5 - 0.48\sin(2\pi t)$ $\qquad H(t) = \dfrac{1}{1.5 - 0.48\sin(2\pi t)}$

Figure V.1 – Trajectories on $(0, 1)$ with $N = 20000$ points, $n(N) = 2042$ points for the estimator $\hat{\alpha}$, and $n(N) = 500$ for $\hat{H}$. α and $\hat{\alpha}$ are represented in the first column, H and $\hat{H}$ in the second column, and in the last column, we have drawn the product $\hat{\alpha}\hat{H}$.

Each function is pretty well-evaluated. We are able to recreate with the estimators the shape of the functions, see for instance Figure V.1. However, for many other examples of functions not displayed here, we notice a significant bias in the estimation of H. It seems to decrease when H is getting values close to 1. We observe this phenomenon with most trajectories, while the estimator $\hat{\alpha}$ seems to be unbiased. We have displayed the product $\hat{\alpha}\hat{H}$ in order to show the link between the estimators. We actually find again the asymtpotic relationship $H(t) = \frac{1}{\alpha(t)}$.

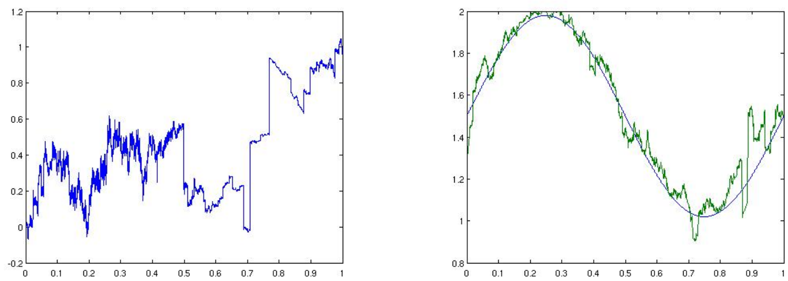
We observe on Figure V.2 an evolution of the variance in the estimation of α. It seems to increase when the function α is decreasing, and we conjecture that the variance at the point t_0 depends on the value $\alpha(t_0)$ in this way. In fact, the increments $Y_{k,N}$ are asymptotically distributed as an $\alpha(t_0)$-stable variable, so we expect that S_N and R_{exp} have a variance increasing when α is decreasing.

We have increased the resolution on Figure V.3, taking more points for the discretization.



a) b) c)

FIGURE V.2 – Trajectory of a Levy process with $\alpha(t) = 1.5+0.48\sin(2\pi t)$ in figure a), and the corresponding estimation of α in figure b) with $n(N) = 2042$. The figure c) represents various estimations of α for the same function $\alpha(t) = 1.5 + 0.48\sin(2\pi t)$, with different trajectories.



d) e)

FIGURE V.3 – Trajectory with $N = 200000$ in figure d), and the estimation with $n(N) = 3546$ in figure e).

The distance observed on Figure V.2.b for α near 1 is then corrected.

V.3.2 Linear multistable multifractional motion

Let $\alpha : \mathbb{R} \to [c, d] \subset (0, 2)$ and $H : \mathbb{R} \to (0, 1)$ be continuously differentiable. Define

$$X(t,u) = C_{\alpha(u)}^{1/\alpha(u)}(\frac{\pi^2 j^2}{3})^{1/\alpha(u)} \sum_{i,j=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)}(|t-V_i|^{H(u)-1/\alpha(u)} - |V_i|^{H(u)-1/\alpha(u)})\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(V_i) \tag{V.2}$$

and the linear multistable multifractional motion

$$Y(t) = X(t,t).$$

Proposition V.5. Assume that $H - \frac{1}{\alpha}$ is a non-negative function, $\lim_{N\to+\infty} n(N) = +\infty$ and $\lim_{N\to+\infty} \frac{N}{n(N)} = +\infty$. Then for all $r > 0$,

$$\lim_{N\to+\infty} \mathsf{E}\left|\hat{H}_N(t_0) - H(t_0)\right|^r = 0 \text{ and } \lim_{N\to+\infty} \mathsf{E}\left|\hat{\alpha}_N(t_0) - \alpha(t_0)\right|^r = 0.$$

Proof

We know from [30] that all the conditions (C1)-(C14) are satisfied. We deduce from Theorem V.1 that $\lim_{N\to+\infty} \mathsf{E}\left|\hat{H}_N(t_0) - H(t_0)\right|^r = 0$. Since the process $X(.,t_0)$ is a $(H(t_0), \alpha(t_0))$ linear fractional stable motion, $X(.,t_0)$ is $H(t_0)$-self-similar with stationary increments [49]. We write $h_{j,t_0}(x) = g(j-x)$ with $g(u) = |1+u|^{H(t_0)-\frac{1}{\alpha(t_0)}} - |u|^{H(t_0)-\frac{1}{\alpha(t_0)}}$ so with Proposition 2.2 of [43], the condition (C*) is satisfied. Let us show that $\lim_{j\to+\infty} \int_{\mathbb{R}} |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} dx = 0$ and conclude with Theorem V.3.

Let $\epsilon > 0$. Let $c_0 > 0$ such that $\int_{|x|>c_0} |h_{0,t_0}(x)|^{\alpha(t_0)} dx \leq \frac{\epsilon}{2}$. Since $\forall x \in [-c_0, c_0]$, $\lim_{j\to+\infty} |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} = 0$ and $(h_{j,t_0}(x))_j$ is uniformly borned on $[-c_0, c_0]$,

$$\lim_{j\to+\infty} \int_{|x|\leq c_0} |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}} dx = 0 \quad \blacksquare$$

We show on Figure V.4 some paths of Lmmm, with the two corresponding estimations of α and H. To simulate the trajectories, we have used the field (V.2). All the increments of $X(.,u)$ are $(H(u), \alpha(u))$-linear fractional stable motions, generated using the LFSN program of [53]. After we have taken the diagonal process $X(t,t)$.

These estimates are overall further than the estimates in the case of the Levy process, because of greater correlations between the increments of the process. However, the estimation of H does not seem to be disturbed by those correlations. The shape of the function H is kept. For α, we notice some disruptions when the function is close to 1. We finally show an example where the estimation of α is not good enough in the last line of Figure V.4. The trajectory, Figure V.4.a), seems to have a big jump, which leads to decrease the estimator $\hat{\alpha}$, represented on Figure V.4.b), while the jump is taken account in the $n(N)$ points. The estimation of H, represented on Figure V.4.c), does not seem to be affected by this phenomenon.

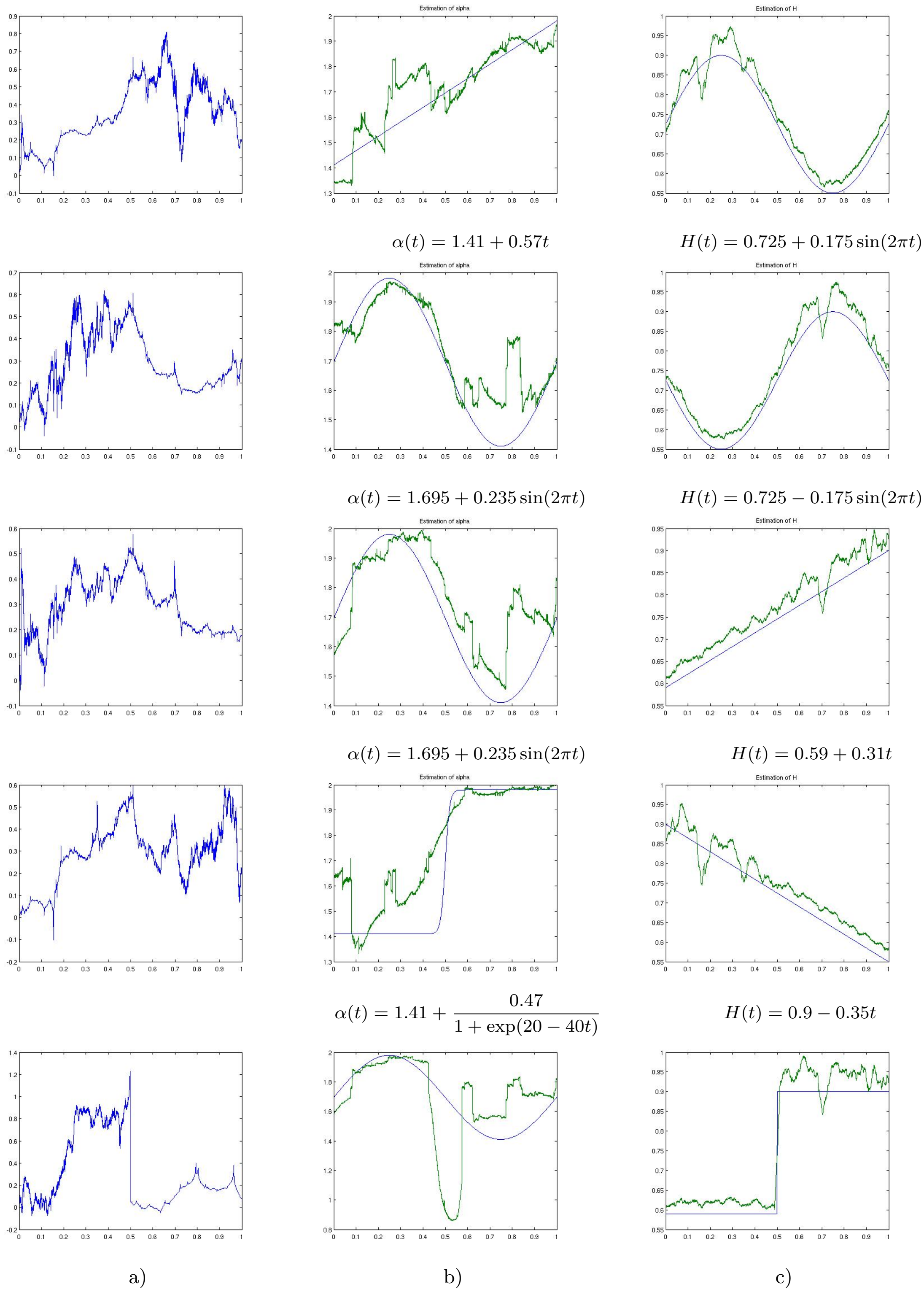


FIGURE V.4 – Trajectories with $N = 20000$ in the first column, the estimations of α with $n(N) = 3000$ points in the second column, and in the last one, the estimations of H with $n(N) = 500$ points.

V.4 Proofs

Proof of Theorem V.1

Note that it is sufficient to prove the result of Theorem V.1 for $r \geq 1$ since the convergence in L^p implies the convergence in L^q for all $q < p$. Let $r \geq 1$. Let H satisfying the condition **(C5)**. We write

$$\begin{aligned}\hat{H}_N(t_0) - H(t_0) &= -\frac{1}{n(N)\log N} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \log \left| \frac{Y_{k,N}}{(\frac{1}{N})^{H(t_0)}} \right| \\ &= -\frac{N}{n(N)\log N} \int_{\frac{[Nt_0]}{N}-\frac{n(N)}{2N}}^{\frac{[Nt_0]}{N}+\frac{n(N)}{2N}} \log \left| \frac{Y(\frac{[Nt]+1}{N}) - Y(\frac{[Nt]}{N})}{(\frac{1}{N})^{H(t_0)}} \right| dt.\end{aligned}$$

Let $\delta_N(dt) = \frac{N}{n(N)} \mathbf{1}_{\{\frac{[Nt_0]}{N}-\frac{n(N)}{2N} \leq t < \frac{[Nt_0]}{N}+\frac{n(N)}{2N}\}} dt$ and $f_N(t) = \log \left| \frac{Y(\frac{[Nt]+1}{N})-Y(\frac{[Nt]}{N})}{(\frac{1}{N})^{H(t)}} \right|$.

Since $\int_0^1 \delta_N(dt) = 1$, we obtain

$$\hat{H}_N(t_0) - H(t_0) = -\frac{1}{\log N} \int_0^1 f_N(t)\delta_N(dt) + \int_0^1 (H(t) - H(t_0))\, \delta_N(dt).$$

Then, there exists a constant $K_r \in \mathbb{R}$ depending on r such that

$$\mathsf{E}\left[|\hat{H}_N(t_0) - H(t_0)|^r\right] \leq K_r \frac{\mathsf{E}\left(|\int_0^1 f_N(t)\delta_N(dt)|^r\right)}{|\log N|^r} + K_r \left| \int_0^1 (H(t) - H(t_0))\, \delta_N(dt) \right|^r.$$

H is continuously differentiable and $\lim_{N\to+\infty} \frac{N}{n(N)} = +\infty$ so

$$\lim_{N\to+\infty} \int_0^1 (H(t) - H(t_0))\, \delta_N(dt) = 0.$$

To conclude, it is sufficient to show that there exists a constant $K \in \mathbb{R}$ depending on t_0 and r such that for all $N \in \mathbb{N}$, $\mathsf{E}\left(|\int_0^1 f_N(t)\delta_N(dt)|^r\right) \leq K$. Let U an open interval satisfying all the conditions **(C1)-(C14)**, and $t_0 \in U$. We can fix $N_0 \in \mathbb{N}$ and $V \subset U$ an open interval depending on t_0 such that for all $N \geq N_0$ and all $t \in V$, $\frac{[Nt]+1}{N} \in U$, $\frac{[Nt]}{N} \in U$ and $\int_0^1 f_N(t)\delta_N(dt) = \int_V f_N(t)\delta_N(dt)$. With the Jensen inequality,

$$\mathsf{E}\left(|\int_0^1 f_N(t)\delta_N(dt)|^r\right) \leq \int_V \mathsf{E}|f_N(t)|^r \delta_N(dt).$$

We consider $\mathsf{E}|f_N(t)|^r = \int\limits_0^{+\infty} \mathsf{P}\left(|f_N(t)|^r > x\right) dx$.

$$\begin{aligned}
\mathsf{E}|f_N(t)|^r &= \int_0^{+\infty} \mathsf{P}\left(f_N(t) > x^{1/r}\right) dx + \int_0^{+\infty} \mathsf{P}\left(f_N(t) < -x^{1/r}\right) dx \\
&= \int_0^{+\infty} \mathsf{P}\left(\left|Y(\frac{[Nt]+1}{N}) - Y(\frac{[Nt]}{N})\right| > \frac{e^{x^{1/r}}}{N^{H(t)}}\right) dx \\
&\quad + \int_0^{+\infty} \mathsf{P}\left(\left|Y(\frac{[Nt]+1}{N}) - Y(\frac{[Nt]}{N})\right| < \frac{e^{-x^{1/r}}}{N^{H(t)}}\right) dx.
\end{aligned}$$

$$\begin{aligned}
\mathsf{P}\left(\left|Y(\frac{[Nt]+1}{N}) - Y(\frac{[Nt]}{N})\right| > \frac{e^{x^{1/r}}}{N^{H(t)}}\right) &\leq \mathsf{P}\left(\left|Y(\frac{[Nt]+1}{N}) - Y(t)\right| \geq \frac{e^{x^{1/r}}}{2N^{H(t)}}\right) \\
&\quad + \mathsf{P}\left(\left|Y(t) - Y(\frac{[Nt]}{N})\right| \geq \frac{e^{x^{1/r}}}{2N^{H(t)}}\right)
\end{aligned}$$

so

$$\mathsf{E}|f_N(t)|^r \leq \mathbf{I}_N^1(t) + \mathbf{I}_N^2(t) + \mathbf{I}_N^3(t)$$

with

$$\mathbf{I}_N^1(t) = \int_0^{+\infty} \mathsf{P}\left(\left|Y(\frac{[Nt]+1}{N}) - Y(t)\right| \geq \frac{e^{x^{1/r}}}{2N^{H(t)}}\right) dx,$$

$$\mathbf{I}_N^2(t) = \int_0^{+\infty} \mathsf{P}\left(\left|Y(\frac{[Nt]}{N}) - Y(t)\right| \geq \frac{e^{x^{1/r}}}{2N^{H(t)}}\right) dx$$

and

$$\mathbf{I}_N^3(t) = \int_0^{+\infty} \mathsf{P}\left(\left|Y(\frac{[Nt]+1}{N}) - Y(\frac{[Nt]}{N})\right| < \frac{e^{-x^{1/r}}}{N^{H(t)}}\right) dx.$$

We consider first $\mathbf{I}_N^1(t)$.

$$\begin{aligned}
\mathbf{I}_N^1(t) &\leq \int_0^{+\infty} \mathsf{P}\left(\left|X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t)\right| \geq \frac{e^{x^{1/r}}}{4N^{H(t)}}\right) dx \\
&\quad + \int_0^{+\infty} \mathsf{P}\left(\left|X(\frac{[Nt]+1}{N}, t) - X(t,t)\right| \geq \frac{e^{x^{1/r}}}{4N^{H(t)}}\right) dx.
\end{aligned}$$

With the conditions **(C1)**, **(C2)** and **(C3)** (or **(C1)**, **(Cs2)**, **(Cs3)** and **(Cs4)** in the σ-finite space case) we can apply Proposition 4.9 or 4.10 of [30] : there exists $K_U > 0$ such that for all $(u, v) \in U^2$ and $x > 0$,

$$\mathsf{P}\left(|X(v,v) - X(v,u)| > x\right) \leq K_U \left(\frac{|v-u|^d}{x^d}(1 + |\log \frac{|v-u|}{x}|^d) + \frac{|v-u|^c}{x^c}(1 + |\log \frac{|v-u|}{x}|^c) \right) \tag{V.3}$$

so there exists $K_U > 0$ such that for all $N \geq N_0$ and all $t \in V$,

$$\begin{aligned} &\mathsf{P}\left(\left| X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t) \right| \geq \frac{e^{x^{1/r}}}{4N^{H(t)}} \right) \\ &\leq K_U \left(\frac{(\log N)^c}{N^{c(1-H(t))} e^{cx^{1/r}}} \right) + K_U \left(\frac{x^{c/r}}{N^{c(1-H(t))} e^{cx^{1/r}}} \right) \\ &+ K_U \left(\frac{(\log N)^d}{N^{d(1-H(t))} e^{dx^{1/r}}} \right) + K_U \left(\frac{x^{d/r}}{N^{d(1-H(t))} e^{dx^{1/r}}} \right). \end{aligned}$$

Since $H_+ < 1$, we conclude that

$$\lim_{N \to +\infty} \int_0^{+\infty} \sup_{t \in U} \mathsf{P}\left(\left| X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t) \right| \geq \frac{e^{x^{1/r}}}{4N^{H(t)}} \right) dx = 0.$$

With the same arguments,

$$\lim_{N \to +\infty} \int_0^{+\infty} \sup_{t \in U} \mathsf{P}\left(\left| X(\frac{[Nt]}{N}, \frac{[Nt]}{N}) - X(\frac{[Nt]}{N}, t) \right| \geq \frac{e^{x^{1/r}}}{4N^{H(t)}} \right) dx = 0. \tag{V.4}$$

Let $\eta < c$. The Markov inequality gives

$$\mathsf{P}\left(\left| X(\frac{[Nt]+1}{N}, t) - X(t,t) \right| \geq \frac{e^{x^{1/r}}}{4N^{H(t)}} \right) \leq \frac{4^\eta N^{\eta H(t)}}{e^{\eta x^{1/r}}} \mathsf{E}\left[|X(\frac{[Nt]+1}{N}, t) - X(t,t)|^\eta \right]$$

and Property 1.2.17 of [49]

$$\mathsf{E}\left[|X(\frac{[Nt]+1}{N}, t) - X(t,t)|^\eta \right] = c_{\alpha(t),0}(\eta)^\eta \left(\int_E (|f(\frac{[Nt]+1}{N}, t, x) - f(t,t,x)|^{\alpha(t)})\, m(dx) \right)^{\eta/\alpha(t)}.$$

With the condition **(C9)**, there exists $K > 0$ such that for all $N \geq N_0$ and all $t \in V$,

$$\int_0^{+\infty} \mathsf{P}\left(\left| X(\frac{[Nt]+1}{N}, t) - X(t,t) \right| \geq \frac{e^{x^{1/r}}}{4N^{H(t)}} \right) dx \leq K.$$

Finally, there exists $K > 0$ such that for all $N \geq N_0$ and all $t \in V$,

$$\mathbf{I}^1_N(t) \leq K.$$

Using the equation (V.4) and the condition **(C9)**, we obtain that there exists $K > 0$ such that for all $N \geq N_0$ and all $t \in V$,

$$\mathbf{I}^2_N(t) \leq K.$$

Thanks to the conditions **(C1)**, **(C6)**, **(C7)**, **(C8)**, **(C10)**, **(C11)**, **(C12)**, **(C13)** and **(C14)**, we conclude for $\mathbf{I}^3_N(t)$ using Proposition 4.11 and 4.8 of [30] : there exists $K > 0$ such that for all $N \geq N_0$ and all $t \in V$,

$$\mathsf{P}\left(\left|Y(\frac{[Nt]+1}{N}) - Y(\frac{[Nt]}{N})\right| < \frac{e^{-x^{1/r}}}{N^{H(t)}}\right) \leq K\frac{e^{-x^{1/r}}}{N^{H(t)}} N^{H(\frac{[Nt]}{N})}$$

so there exists $K > 0$ such that for all $N \geq N_0$ and all $t \in V$,

$$\mathbf{I}^3_N(t) \leq K \quad \blacksquare$$

Proof of Theorem V.2
Let $p \in [p_0, \alpha(t_0))$. We define

$$A_N(p) = \frac{N^{pH(t_0)}}{n(N)} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \left|X(\frac{k+1}{N}, \frac{k+1}{N}) - X(\frac{k+1}{N}, t_0)\right|^p,$$

$$B_N(p) = \frac{N^{pH(t_0)}}{n(N)} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \left|X(\frac{k}{N}, \frac{k}{N}) - X(\frac{k}{N}, t_0)\right|^p$$

and

$$C_N(p) = \frac{N^{pH(t_0)}}{n(N)} \sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} \left|X(\frac{k+1}{N}, t_0) - X(\frac{k}{N}, t_0)\right|^p.$$

We have, for $p \leq 1$,

$$\begin{aligned} \mathsf{P}\left(|N^{pH(t_0)}S^p_N(p) - \mathsf{E}|Z|^p| > x\right) &\leq \mathsf{P}\left(|N^{pH(t_0)}S^p_N(p) - C_N(p)| \geq \frac{x}{2}\right) + \mathsf{P}\left(|\mathsf{E}|Z|^p - C_N(p)| \geq \frac{x}{2}\right) \\ &\leq \mathsf{P}\left(|\mathsf{E}|Z|^p - C_N(p)| \geq \frac{x}{2}\right) + \mathsf{P}\left(A_N(p) + B_N(p) \geq \frac{x}{2}\right) \end{aligned}$$

and for $p \geq 1$,

$$\begin{aligned} \mathsf{P}\left(|N^{H(t_0)}S_N(p) - (\mathsf{E}|Z|^p)^{\frac{1}{p}}| > x\right) &\leq \mathsf{P}\left(|N^{H(t_0)}S_N(p) - C^{\frac{1}{p}}_N(p)| \geq \frac{x}{2}\right) \\ &\quad + \mathsf{P}\left(|C^{\frac{1}{p}}_N(p) - (\mathsf{E}|Z|^p)^{\frac{1}{p}}| \geq \frac{x}{2}\right) \\ &\leq \mathsf{P}\left(|(\mathsf{E}|Z|^p)^{\frac{1}{p}} - C^{\frac{1}{p}}_N(p)| \geq \frac{x}{2}\right) + \mathsf{P}\left(A^{\frac{1}{p}}_N(p) + B^{\frac{1}{p}}_N(p) \geq \frac{x}{2}\right). \end{aligned}$$

To prove Theorem V.2, it is enough to show that $A_N(p) \xrightarrow{\mathsf{P}} 0$, $B_N(p) \xrightarrow{\mathsf{P}} 0$ and $C_N(p) \xrightarrow{\mathsf{P}} \mathsf{E}|Z|^p$, with $Z \sim S_{\alpha(t_0)}(1,0,0)$.

We consider first $A_N(p) \xrightarrow{\mathsf{P}} 0$. Let $\delta_N(dt) = \frac{N}{n(N)} \mathbf{1}_{\{\frac{[Nt_0]}{N} - \frac{n(N)}{2N} \le t < \frac{[Nt_0]}{N} + \frac{n(N)}{2N}\}} dt$. Let U an open interval satisfying the conditions of the theorem and $t_0 \in U$. We can fix $N_0 \in \mathbb{N}$ and $V \subset U$ an open interval depending on t_0 such that for all $N \geq N_0$ and all $t \in V$, $\frac{[Nt]+1}{N} \in U$, $\frac{[Nt]}{N} \in U$, $\int_0^1 \delta_N(dt) = \int_V \delta_N(dt)$, and such that the inequality (V.3) holds.

$$\begin{aligned}
\mathsf{P}\left(A_N(p) > x\right) &= \mathsf{P}\left(\int_0^1 \left|\frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}\right|^p \delta_N(dt) > x\right) \\
&\leq \frac{1}{x}\int_0^1 \mathsf{E}\left[\left|\frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}\right|^p\right] \delta_N(dt) \\
&= \frac{1}{x}\int_V \mathsf{E}\left[\left|\frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}\right|^p\right] \delta_N(dt)
\end{aligned}$$

Let $t \in V$.

$$\mathsf{E}\left[\left|\frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}\right|^p\right] = \int_0^\infty \mathsf{P}\left(\left|\frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}\right| > u^{1/p}\right) du.$$

Let $u > 0$. We know from (V.3) that there exists $K_U > 0$ such that for all $t \in V$,

$$\mathsf{P}\left(\left|\frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}\right| > u^{1/p}\right) \leq K_U \frac{((\log N)^c + |\log u|^c)}{N^{c(1-H(t_0))} u^{c/p}} + K_U \frac{((\log N)^d + |\log u|^d)}{N^{d(1-H(t_0))} u^{d/p}},$$

so, with the assumption $H(t_0) < 1$,

$$\lim_{N \to +\infty} \mathsf{P}\left(\left|\frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}\right| > u^{1/p}\right) = 0.$$

There exists $K_{U,p} > 0$ such that

$$\mathsf{P}\left(\left|\frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}\right| > u^{1/p}\right) \leq \mathbf{1}_{u<1} + K_{U,p}\left(\frac{|\log u|^d}{u^{d/p}} + \frac{|\log u|^c}{u^{c/p}}\right) \mathbf{1}_{u \geq 1}. \tag{V.5}$$

Since α is a continuous function, we can fix U small enough such that $c = \inf_{t \in U} \alpha(t) > p$. We deduce from the dominated convergence theorem that for all $t \in U$,

$$\lim_{N \to +\infty} \mathsf{E}\left[\left|\frac{X(\frac{[Nt]+1}{N}, \frac{[Nt]+1}{N}) - X(\frac{[Nt]+1}{N}, t_0)}{(1/N)^{H(t_0)}}\right|^p\right] = 0.$$

With the inequality (V.5),

$$\mathsf{E}\left[\left|\frac{X(\frac{[Nt]+1}{N},\frac{[Nt]+1}{N})-X(\frac{[Nt]+1}{N},t_0)}{(1/N)^{H(t_0)}}\right|^p\right]\leq 1+\int_1^{+\infty}K_{U,p}\left(\frac{|\log u|^d}{u^{d/p}}+\frac{|\log u|^c}{u^{c/p}}\right)du$$

and again with the dominated convergence theorem,

$$\lim_{N\to+\infty}\mathsf{P}\left(A_N(p)>x\right)=0.$$

The same inequalities holds with $B_N(p)$ so we obtain $B_N(p)\xrightarrow{\mathsf{P}}0$. We conclude proving $C_N(p)\xrightarrow{\mathsf{P}}\mathsf{E}|Z|^p$. Let $c_0>0$. We use the decomposition

$$C_N(p)-\mathsf{E}|Z|^p=\frac{1}{n(N)}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}\left|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|^p\mathbf{1}_{\left|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|>c_0}-\mathsf{E}|Z|^p\mathbf{1}_{|Z|>c_0}$$

$$+\frac{1}{n(N)}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}\left|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|^p\mathbf{1}_{\left|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|\leq c_0}-\mathsf{E}|Z|^p\mathbf{1}_{|Z|\leq c_0}.$$

Let $\epsilon>0$ and $x>0$. By Markov's inequality, we have

$$\begin{aligned}\mathsf{P}_1&=\mathsf{P}\left(\frac{1}{n(N)}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}\left|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|^p\mathbf{1}_{\left|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|>c_0}>\frac{x}{4}\right)\\&\leq\frac{4}{xn(N)}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}\mathsf{E}\left[\left|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|^p\mathbf{1}_{\left|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|>c_0}\right].\end{aligned}$$

Since $X(.,t_0)$ is $H(t_0)$-self-similar with stationary increments,

$$\mathsf{P}_1\leq\frac{4}{x}\mathsf{E}\left[|X(1,t_0)|^p\,\mathbf{1}_{|X(1,t_0)|>c_0}\right]$$

and

$$\mathsf{E}|Z|^p\mathbf{1}_{|Z|\leq c_0}=\frac{1}{n(N)}\sum_{k=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1}\mathsf{E}\left[\left|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|^p\mathbf{1}_{\left|\frac{X(\frac{k+1}{N},t_0)-X(\frac{k}{N},t_0)}{(1/N)^{H(t_0)}}\right|\leq c_0}\right].$$

We fix c_0 large enough such that for all $N\in\mathbb{N}$, $\mathsf{P}_1\leq\frac{\epsilon}{2}$ and $\mathsf{E}|Z|^p\mathbf{1}_{|Z|>c_0}<\frac{x}{4}$. Writing $K(x)=|x|^p\mathbf{1}_{|x|\leq c_0}$ and $\Delta X_{k,t_0}=X(k+1,t_0)-X(k,t_0)$, using Chebyshev's inequality, we get

$$\begin{aligned}
\mathsf{P}\left(|C_N(p)-\mathsf{E}|Z|^p|>x\right) &\leq \frac{\epsilon}{2}+\frac{4}{x^2 n(N)^2}\sum_{k,j=[Nt_0]-\frac{n(N)}{2}}^{[Nt_0]+\frac{n(N)}{2}-1} Cov\left(K(\Delta X_{k,t_0}),K(\Delta X_{j,t_0})\right)\\
&\leq \frac{\epsilon}{2}+\frac{4}{x^2}\frac{Var\left(K(\Delta X_{0,t_0})\right)}{n(N)}+\frac{4}{x^2}\frac{1}{n(N)}\sum_{j=1}^{n(N)-1} Cov\left(K(\Delta X_{0,t_0}),K(\Delta X_{j,t_0})\right).
\end{aligned}$$

Under the condition **(C*)**, we can apply Theorem 2.1 of [43] : there exists a positive constant C such that

$$Cov\left(K(\Delta X_{0,t_0}),K(\Delta X_{j,t_0})\right)\leq C\|K\|_1^2\int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}}m(dx).$$

Since the process $X(.,t_0)$ is $H(t_0)$-self-similar with stationary increments, the constant C does not depend on k, j. We then obtain the existence of a positive constant C_{p,c_0} depending on p, c_0 and x such that

$$\mathsf{P}\left(|C_N(p)-\mathsf{E}|Z|^p|>x\right)\leq \frac{\epsilon}{2}+\frac{C_{p,c_0}}{n(N)}\int_E |h_{0,t_0}(x)|^{\alpha(t_0)}m(dx)+\frac{C_{p,c_0}}{n(N)}\sum_{j=1}^{n(N)-1}\int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}}m(dx).$$

Since $\lim\limits_{N\to+\infty} n(N)=+\infty$ and $\lim\limits_{j\to+\infty}\int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}}m(dx)=0$, we conclude with Cesaro's theorem that there exists $N_0\in\mathbb{N}$ such that for all $N\geq N_0$,

$$\frac{C_{p,c_0}}{n(N)}\int_E |h_{0,t_0}(x)|^{\alpha(t_0)}m(dx)+\frac{C_{p,c_0}}{n(N)}\sum_{j=1}^{n(N)-1}\int_E |h_{0,t_0}(x)h_{j,t_0}(x)|^{\frac{\alpha(t_0)}{2}}m(dx)\leq\frac{\epsilon}{2}$$

and

$$\mathsf{P}\left(|C_N(p)-\mathsf{E}|Z|^p|>x\right)\leq\epsilon \quad \blacksquare$$

Proof of Theorem V.3

Since $x\to x^\gamma$ is an increasing function on $\mathbb{R}_+$,

$$\hat{\alpha}_N(t_0)=\min\left(\underset{\alpha\in[0,2]}{\arg\min}\int_{p_0}^2 |R_{\exp}(p)-R_\alpha(p)|^\gamma dp\right).$$

Let $g_N(\alpha)=\int\limits_{p_0}^{2}|R_{\exp}(p)-R_\alpha(p)|^\gamma dp$ and $g(\alpha)=\int\limits_{p_0}^{2}|R_{\alpha(t_0)}(p)-R_\alpha(p)|^\gamma dp$.

g is a continuous function on $(0,2]$, with $g(0)>0$, $g(2)>0$. The only solution of the equation $g(\alpha)=0$ is $\alpha(t_0)$. Moreover, $\lim\limits_{\alpha\to\alpha(t_0)}\frac{|g(\alpha)-g(\alpha(t_0))|}{|\alpha-\alpha(t_0)|^\gamma}>0$.

Then, there exists $K_{\alpha(t_0)}$ a positive constant depending only on $\alpha(t_0)$ such that :

$$\forall\alpha\in(0,2),\quad |g(\alpha)|\geq K_{\alpha(t_0)}|\alpha-\alpha(t_0)|. \tag{V.6}$$

We estimate now $|g(\hat{\alpha}_N(t_0))|$.

$$\begin{aligned}|g(\hat{\alpha}_N(t_0))| &\leq |g(\hat{\alpha}_N(t_0)) - g_N(\hat{\alpha}_N(t_0))| + |g_N(\hat{\alpha}_N(t_0))| \\ &\leq |g(\hat{\alpha}_N(t_0)) - g_N(\hat{\alpha}_N(t_0))| + g_N(\alpha(t_0)),\end{aligned}$$

and

$$\begin{aligned}|g(\hat{\alpha}_N(t_0)) - g_N(\hat{\alpha}_N(t_0))| &= \left|\int_{p_0}^{2} \left(|R_{\alpha(t_0)}(p) - R_{\hat{\alpha}_N(t_0)}(p)|^\gamma - |R_{\exp}(p) - R_{\hat{\alpha}_N(t_0)}(p)|^\gamma\right) dp\right| \\ &\leq \int_{p_0}^{2} \left|R_{\alpha(t_0)}(p) - R_{\exp}(p)\right|^\gamma dp \\ &= g_N(\alpha(t_0)).\end{aligned}$$

From (V.6),

$$\begin{aligned}|\hat{\alpha}_N(t_0) - \alpha(t_0)| &\leq \frac{1}{K_{\alpha(t_0)}} g(\hat{\alpha}_N(t_0)) \\ &\leq \frac{2}{K_{\alpha(t_0)}} g_N(\alpha(t_0)).\end{aligned}$$

Let us show that $\forall r > 0$, $\lim\limits_{N\to+\infty} \mathsf{E}\,|g_N(\alpha(t_0))|^r = 0$. Let $r > 0$. One has, using the inequality $S_N(p) \leq S_N(q)$ for $p \leq q$,

$$\begin{aligned}g_N(\alpha(t_0)) &= \int_{p_0}^{\alpha(t_0)} |R_{\exp}(p) - R_{\alpha(t_0)}(p)|^\gamma dp + \int_{\alpha(t_0)}^{2} |R_{\exp}(p)|^\gamma dp \\ &\leq \int_{p_0}^{\alpha(t_0)} |R_{\exp}(p) - R_{\alpha(t_0)}(p)|^\gamma dp + (2 - \alpha(t_0)) \left|\frac{S_N(p_0)}{S_N(\alpha(t_0))}\right|^\gamma.\end{aligned}$$

For the first term, we use Theorem V.2 : for all $p \in [p_0, \alpha(t_0))$,

$$N^{H(t_0)} S_N(p) \xrightarrow{\mathsf{P}} (\mathsf{E}|Z|^p)^{1/p} \tag{V.7}$$

where $Z \sim S_{\alpha(t_0)}(1,0,0)$. It is clear that $\forall p \in [p_0, \alpha(t_0))$,

$$\left(N^{H(t_0)} S_N(p_0), N^{H(t_0)} S_N(p)\right) \xrightarrow{\mathsf{P}} \left((\mathsf{E}|Z|^{p_0})^{1/p_0}, (\mathsf{E}|Z|^p)^{1/p}\right),$$

and

$$R_{\exp}(p) = \frac{S_N(p_0)}{S_N(p)} \xrightarrow{\mathsf{P}} R_{\alpha(t_0)}(p). \tag{V.8}$$

Note that $\forall N \in \mathbb{N}$, $\forall p \in [p_0, \alpha(t_0))$, $|R_{\exp}(p)| \leq 1$ so there exists a positive constant K depending on γr, $\alpha(t_0)$ and p such that

$$\mathsf{E}|R_{\exp}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} \quad = \quad \int_0^K \mathsf{P}\left(|R_{\exp}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} > x\right) dx.$$

Finally, with (V.8), $\forall p \in [p_0, \alpha(t_0))$, $\mathsf{E}|R_{\exp}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} \underset{N\to+\infty}{\longrightarrow} 0$. With the inequality $\mathsf{E}|R_{\exp}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} \leq 2C_{\gamma r}$ where $C_{\gamma r}$ is a positive constant depending on γr, by the dominating convergence theorem,

$$\lim_{N\to+\infty} \int_{p_0}^{\alpha(t_0)} \mathsf{E}|R_{\exp}(p) - R_{\alpha(t_0)}(p)|^{\gamma r} dp = 0.$$

To conclude we show that $\left|\frac{S_N(p_0)}{S_N(\alpha(t_0))}\right|^\gamma \xrightarrow{L^r} 0$. Since $\forall N \in \mathbb{N}$, $\left|\frac{S_N(p_0)}{S_N(\alpha(t_0))}\right|^\gamma \leq 1$, it is enough to show $\frac{S_N(p_0)}{S_N(\alpha(t_0))} \xrightarrow{\mathsf{P}} 0$. Let $p < \alpha(t_0)$.

$$\mathsf{P}(\frac{1}{|N^{H(t_0)} S_N(\alpha(t_0))|} > x) \leq \mathsf{P}(\frac{1}{|N^{H(t_0)} S_N(p)|} > x).$$

So,

$$\begin{aligned}
\limsup_{N\to+\infty} \mathsf{P}(\frac{1}{|N^{H(t_0)} S_N(\alpha(t_0))|} > x) \quad &\leq \quad \limsup_{N\to+\infty} \mathsf{P}(\frac{1}{|N^{H(t_0)} S_N(p)|} > x) \\
&= \quad \lim_{N\to+\infty} \mathsf{P}(\frac{1}{|N^{H(t_0)} S_N(p)|} > x) \\
&= \quad \mathsf{P}(\frac{1}{(\mathsf{E}|Z|^p)^{1/p}} > x),
\end{aligned}$$

with (V.7). Since $\lim\limits_{p\to\alpha(t_0)} \mathsf{P}(\frac{1}{(\mathsf{E}|Z|^p)^{1/p}} > x) = 0$, we have $\limsup\limits_{N\to+\infty} \mathsf{P}(\frac{1}{|N^{H(t_0)} S_N(\alpha(t_0))|} > x) = 0$ and $\frac{1}{N^{H(t_0)} S_N(\alpha(t_0))} \xrightarrow{\mathsf{P}} 0$. Using the convergence $N^{H(t_0)} S_N(p_0) \xrightarrow{\mathsf{P}} (\mathsf{E}|Z|^{p_0})^{1/p_0}$, we obtain $\frac{S_N(p_0)}{S_N(\alpha(t_0))} \xrightarrow{\mathsf{P}} 0$ ■

V.5 Assumptions

This section gathers the various conditions required on the considered processes so that our results hold. For all the assumptions, we shall denote $c = \inf_{v\in U} \alpha(v)$ and $d = \sup_{v\in U} \alpha(v)$.

- (C1) The family of functions $v \to f(t, v, x)$ is differentiable for all (v, t) in U^2 and almost all x in E. The derivatives of f with respect to v are denoted by f'_v.
- (C2) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t\in U} \int_{\mathbb{R}} \left[\sup_{w\in U} (|f(t, w, x)|^{\alpha(w)})\right]^{1+\delta} \hat{m}(dx) < \infty.$$

- (Cs2) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbb{R}} \left[\sup_{w \in U} (|f(t,w,x)|^{\alpha(w)}) \right]^{1+\delta} r(x)^{\delta} \, m(dx) < \infty.$$

- (C3) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbb{R}} \left[\sup_{w \in U} (|f'_v(t,w,x)|^{\alpha(w)}) \right]^{1+\delta} \hat{m}(dx) < \infty.$$

- (Cs3) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbb{R}} \left[\sup_{w \in U} (|f'_v(t,w,x)|^{\alpha(w)}) \right]^{1+\delta} r(x)^{\delta} \, m(dx) < \infty.$$

- (Cs4) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbb{R}} \left[\sup_{w \in U} \left[|f(t,w,x) \log(r(x))|^{\alpha(w)} \right] \right]^{1+\delta} r(x)^{\delta} \, m(dx) < \infty.$$

- (C5) $X(t,u)$ (as a process in t) is localisable at u with exponent $H(u) \in (H_-, H_+) \subset (0,1)$, with local form $X'_u(t,u)$, and $u \mapsto H(u)$ is a C^1 function .
- (C6) There exists $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbb{R}$,

$$|f(v,u,x)| \leq K_U.$$

- (C7) There exists $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbb{R}$,

$$\left| f'_v(v,u,x) \right| \leq K_U.$$

- (C8) There exists $K_U > 0$ and a function H defined on U such that $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbb{R}$,

$$\frac{1}{|v-u|^{H(u)-1/\alpha(u)}} \, |f(v,u,x) - f(u,u,x)| \leq K_U.$$

- (C9) There exists $\varepsilon_0 > 0$, $K_U > 0$ and a function H defined on U such that $\forall r < \varepsilon_0$, $\forall t \in U$,

$$\frac{1}{r^{H(t)\alpha(t)}} \int_{\mathbb{R}} |f(t+r,t,x) - f(t,t,x)|^{\alpha(t)} m(dx) \leq K_U.$$

- (C10) There exists $p \in (d,2)$, $p \geq 1$, $K_U > 0$ and a function H defined on U such that $\forall v \in U$,$\forall u \in U$,

$$\frac{1}{|v-u|^{1+p(H(u)-\frac{1}{\alpha(u)})}} \int_{\mathbb{R}} |f(v,u,x) - f(u,u,x)|^p \, m(dx) \leq K_U.$$

- (C11) There exists $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$,

$$\int_{\mathbb{R}} |f(v,u,x)|^2 \, m(dx) \leq K_U.$$

- (C12)
$$\inf_{v\in U}\int_{\mathbb{R}} f(v,v,x)^2 m(dx) > 0.$$
- (C13) There exists a positive function g and a function H defined on U such that
$$\lim_{r\to 0}\sup_{t\in U}\left|\frac{1}{r^{1+2(H(t)-1/\alpha(t))}}\int_{\mathbb{R}}\left(f(t+r,t,x)-f(t,t,x)\right)^2 m(dx) - g(t)\right| = 0.$$
- (C14) $\exists K_U > 0$ such that, $\forall v \in U$, $\forall u \in U$,
$$\frac{1}{|v-u|^2}\int_{\mathbb{R}}\left|f(v,v,x)-f(v,u,x)\right|^2 m(dx) \le K_U.$$

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Table des figures

Résumé : Nous étudions les propriétés probabilistes, trajectorielles et statistiques des processus stochastiques multistables, qui sont tangents en chaque point à un processus stable. Ils possèdent ainsi une intensité de sauts et une régularité locale qui varient au cours du temps. Nous nous intéressons dans un premier temps aux processus pouvant être définis par une moyenne mobile et possédant la propriété d'être localisables, c'est-à-dire d'être tangents en loi à un processus en chaque point. Des critères assurant la localisabilité, ainsi qu'une méthode de simulation de tels processus sont donnés. Nous proposons ensuite une nouvelle construction et des critères de localisabilité des processus multistables à l'aide d'une représentation de type Ferguson-Klass-LePage. Pour les processus obtenus, nous étudions certaines propriétés probabilistes et trajectorielles. En particulier, nous caractérisons le comportement asymptotique des accroissements des processus multistables, ainsi que leur régularité Höldérienne. Enfin, nous proposons des estimateurs de la fonction de stabilité et de la fonction de localisabilité. La consistance au sens de la convergence L^p est prouvée. Les performances des estimateurs sont illustrées sur des séries simulées suivant deux modèles : le mouvement de Lévy multistable et le mouvement linéaire multifractionnaire multistable.

Mots clés : Processus stables - Processus ponctuels - Processus de sauts - Représentation de Ferguson-Klass-LePage - Régularité locale - Exposant de Hölder - Théorème limite dans L^p - Estimation fonctionnelle.

Summary : This PhD thesis deals with some probabilistic, pathwise and statistical properties of multistable stochastic processes, which are tangent at any point to a stable process. Their intensity of jumps and their local regularity are varying with time. We first consider the processes possibly defined as a moving average which are localisable, that is they are tangent to a non-trivial process at any point. We give general conditions which ensure that the moving average is localisable and we characterize the nature of the associated tangent process. We also consider the problem of path synthesis, for which we give both theoretical results and numerical simulations. We present then a different construction of the multistable processes, based on the Ferguson-Klass-LePage series representation. We consider various particular cases of interest, including multistable Levy motion and linear multistable multifractional motion. We study then some probabilistic properties. In particular, we describe the behavior of the incremental moments and the pointwise Hölder exponent. We compute the precise value of the almost sure Hölder exponent in the case of the multistable Levy motion. Finally, we give two estimators of the stability and the localisability functions, and we prove the consistency of those two estimators. We illustrate these convergences with the Levy multistable process and the Linear Multifractional Multistable Motion.

Key words : Stable processes - Pointwise processes - Jump processes - Ferguson-Klass-LePage representation - Local regularity - Hölder exponent - Limit theorem in L^p - Functional estimation.